Gradient expansion approach to multiple-band Fermi liquids

Ryuichi Shindou* and Leon Balents
Department of Physics, University of California, Santa Barbara, California 93106, USA
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Promoted by the recent progress of Berry phase physics in spin galvanomagnetic communities, we develop a systematic derivation of the reduced Keldysh equation (RKE) which captures the low-energy dynamics of quasiparticles constrained within doubly degenerate bands forming a single Fermi surface. The derivation begins with the Keldysh equation for a quite general multiple-band interacting Fermi system, which is originally an $N_b \times N_b$ matrix-formed integral (or infinite-order differential) equation, with N_b being the total number of bands. To derive the RKE for quasiparticle on a Fermi surface in question, we project out both the fully occupied and empty band degrees of freedom perturbatively in the gradient expansion, whose coupling constant measures how a system is disequilibrated. As for the electron-electron interactions, however, we only employ the so-called adiabatic assumption of the Fermi liquid theory, so that the electron correlation effects onto the adiabatic transport of quasiparticles, i.e., the Hermitian (real) part of the self-energy, are taken into account in an unbiased manner. The RKE thus derived becomes an SU(2) *covariant* differential equation and treats the spin and charge degrees of freedom on an equal footing. Namely, the quasiparticle spin precessions due to the non-Abelian gauge fields are automatically encoded into its covariant derivatives. When further solved in favor of spectral functions, this covariant differential equation suggests that quasiparticles on a doubly degenerate Fermi surface acquire *spin-selective* Berry curvature corrections under the applied electromagnetic fields. This theoretical observation gives us some hints of possible experimental methodology for measuring the SU(2) Berry's curvatures by spin-resolved photoemission experiments. Due to the nontrivial frequency dependence of (the Hermitian part of) self-energy, our RKE is composed of Berry's curvatures in the $d+1$ dual space, i.e., k-ω space, so that the dual electric field is already introduced. To provide a simple way to understand this "temporal" component of the U(1) Berry's curvature, we also provide the dual analog of the Ampere's law, where the "spatial" rotation of the electric field in combination with the temporal derivative of the well-known magnetic component is determined by the U(1) magnetic monopole "current."

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I. INTRODUCTION

Gauge fields often appear in effective low-energy theories in condensed matter physics, whenever the system's low-energy manifold is restricted by some "local" constraints.[1] Classic examples include doped Mott-Hubbard insulators, two-dimensional electron gases in the fractional quantum Hall regime, low-dimensional quantum spin systems, and highly frustrated magnets.

In these "strong-coupling" problems, the constraints are implemented locally in *real space*. For example, the on-site Coulomb interaction in the Hubbard model or t-J model forbids double occupancy on *every* site. In strong magnetic fields, the wave function for electrons in a two-dimensional electron gas must be annihilated by the Landau level lowering operator, which is a differential operator defined locally for *each* value of the two-dimensional coordinate. In the quantum dimer model, spins are presumed to be bound into singlets which cover *each* site exactly once. In many cases, these local constraints take the form of a version of "Gauss' law" imposed on suitably introduced "electromagnetic field" and "charge" variables. Due to the Gauss' law constraint, the low-energy effective theory for such systems becomes mathematically equivalent to a form of (usually compact) quantum electrodynamics.[1–4]

Another kind of constrained system occurs in the weak-coupling region, such as in a Fermi liquid (FL), where the states at the Fermi surface are supposed to be well separated from the other fully occupied and empty bands by a sufficiently large (direct) band gap. The low-energy manifold is thus spanned only by the Bloch states of those conduction bands forming this Fermi surface. Even in the presence of strong electron correlations, we may still assume that the low-energy Hilbert space is spanned only by those quasiparticle excitations constrained within a Fermi surface, as far as electron-electron interactions can be introduced adiabatically in comparison with the direct band gap.

In most of the literature, Fermi liquids are assumed to be well described by ignoring electron correlations completely. In such a noninteracting case, the projection to a single low-energy band or degenerate bands has been studied extensively,[5–10] and yields an effective equation of motion (EOM) for the conduction electrons in this band, say, the αth band, moving under the influence of *external* electric field $\mathbf{e}$ and magnetic field $\mathbf{b}$. Such an EOM contains a *dual magnetic field* $\mathcal{B}^\alpha$, which acts on a quasiparticle like a *Lorentz force in k-space*:

$$\frac{dR}{dT} = \mathbf{z}^\dagger \left\{ \hat{\mathbf{v}}_\alpha + \mathcal{B}^\alpha \times \frac{dk}{dT} \right\} \mathbf{z}, \quad \frac{dk}{dT} = -\mathbf{e} + \mathbf{b} \times \frac{dR}{dT},$$

$$i\frac{d\mathbf{z}}{dT} = \left\{ \mathbf{M}_\alpha \cdot \mathbf{b} + \sum_i^d \mathcal{A}^\alpha \cdot \frac{dk}{dT} \right\} \mathbf{z}. \tag{1}$$

$\mathbf{z}$ is a CP^{N-1} vector in FLs with N-fold degenerate conduction bands, i.e., $\mathbf{z}=(z_1, z_2, \ldots, z_N)$ with $\mathbf{z}^\dagger \mathbf{z}=1$. This complex-

valued vector describes the internal degrees of freedom associated with the degeneracy at each k point. Correspondingly, the dual magnetic field, magnetic gauge field $\mathcal{A}^\alpha$, and "orbital magnetization" $\mathbf{M}_\alpha$ are all $N\times N$ Hermitian matrices. In a noninteracting Fermi gas, they are defined solely in terms of the periodic part of Bloch wave functions of conduction bands $|u_\sigma\rangle$ $(\sigma=1,\ldots,N\in\alpha)$:

$$\mathcal{B}_i^\alpha \equiv i\epsilon_{ijm}\partial_{k_j}\mathcal{A}_m^\alpha + i\epsilon_{ijm}\mathcal{A}_j^\alpha\mathcal{A}_m^\alpha, \tag{2}$$

$$[\mathcal{A}_j^\alpha]_{(\sigma|\sigma')} \equiv \langle u^{\alpha\sigma}|\partial_{k_j}u^{\alpha\sigma'}\rangle, \tag{3}$$

$$[\mathbf{M}_{\alpha,m}]_{(\sigma|\sigma')} \equiv \frac{i\epsilon_{mnl}}{2}\langle\partial_{k_n}u^{\alpha\sigma}|\hat{H}-E_\alpha|\partial_{k_l}u^{\alpha\sigma'}\rangle, \tag{4}$$

with E_α being the energy dispersion for the αth band. Especially, the third EOM dictates that the CP^{N-1} vector $\mathbf{z}$ precesses due to the nontrivial matrix structures of $[\mathbf{M}_\alpha]$ (Zeeman field) and $[\mathcal{A}_j^\alpha]$ (Wilczek-Zee phase), respectively.

One purpose of this paper is to enlarge the regime of validity of this effective EOM, so as to include metals for which electron-electron interactions are significant, i.e., Fermi *liquids* rather than Fermi *gases*. The SU(N) effective EOM mentioned above is clearly valid *only* for a *noninteracting* Fermi gas, or within a mean-field description of ordered states such as (ferro)magnetic metals in which the *ordered moment does not fluctuate at all*. The fundamental framework of Fermi liquid theory, however, implies that these effective EOMs should be properly generalized into a realistic metal, where the electron-electron interaction and magnetic fluctuations are not weak. In fact, Haldane has recently argued that the "renormalized" Bloch wave function for a quasiparticle should be taken as the eigenvector of spectral function.[11,12] In this paper, we provide a quite general derivation of the effective EOM for quasiparticles based on the Keldysh formalism, which confirms in part that this notion is valid in an arbitrary U(1) Fermi liquid.

For $N=1$ case, the *form* of this EOM is identical to Eq. (1). However, in passing from the noninteracting version of gauge field definition, i.e., Eq. (3), to that of the many-body case, we encounter an additional complication. Because of the energy dependence of the self-energy, the renormalized Bloch wave function for quasiparticles—defined from the Green's function—depends on the energy or frequency ω, in addition to crystal momentum k, i.e., $|u^\alpha(k,\omega)\rangle$. Accordingly, we are naturally led to introduce a sort of dual version of the electric field and associated electrostatic potential:

$$\mathcal{E}_j^\alpha \equiv i(\partial_\omega\mathcal{A}_j^\alpha - \partial_{k_j}\mathcal{A}_0^\alpha) + i[\mathcal{A}_0^\alpha,\mathcal{A}_j^\alpha], \tag{5}$$

$$[\mathcal{A}_0^\alpha]_{(\sigma|\sigma')} \equiv \langle u^{\alpha\sigma}|\partial_\omega u^{\alpha\sigma'}\rangle, \tag{6}$$

which, as well as the magnetic component, generally have nontrivial structures in the $(d+1)$-dimensional dual space.

As will be shown in this paper, the renormalized k-space Lorentz force appearing in the effective EOM for quasiparticles is composed both of these magnetic and electric components estimated on shell:

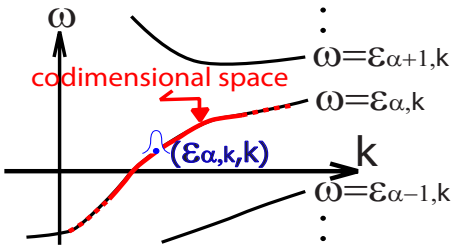

FIG. 1. (Color online) ω-k Euclidean space and its codimensional subspace associated with $\omega=\epsilon_{\alpha,k}$ (red line). $\tilde{\mathcal{B}}^\alpha$ can be viewed as just the magnetic component of Berry's curvature defined in latter subspace.

$$\tilde{\mathcal{B}}^\alpha \equiv \bar{\mathcal{B}}^\alpha - \bar{\mathcal{E}}^\alpha\times\mathbf{v}_\alpha, \tag{7}$$

$$\bar{\mathcal{B}}^\alpha \equiv (\mathcal{B}^\alpha)_{|\omega=\epsilon_{\alpha,k}}, \quad \bar{\mathcal{E}}^\alpha \equiv (\mathcal{E}^\alpha)_{|\omega=\epsilon_{\alpha,k}}, \quad \mathbf{v}_{\alpha,j} = \frac{\partial\epsilon_{\alpha,k}}{\partial k_j}. \tag{8}$$

$\epsilon_{\alpha,k}$ is the renormalized energy dispersion for a quasiparticle in question, which is defined as the pole of the single-point Green's function with respect to ω [see Eq. (23)]. In this sense, this newly introduced electric component *provides* another source of the Lorentz force in the k space, particular only to *interacting* Fermi systems.

From the viewpoint of this EOM alone, however, the significance of the temporal component of the Berry's curvature is not clear. Namely, the renormalized Lorentz field $\tilde{\mathcal{B}}^\alpha$ can be also viewed as just the "magnetic component" defined in the codimensional space associated with the quasiparticle energy dispersion $\omega=\epsilon_{\alpha,k}$,

$$\tilde{\mathcal{B}}_j^\alpha \equiv i\epsilon_{jml}\partial_{k_m}\tilde{\mathcal{A}}_{k_l}^\alpha + i\epsilon_{jml}\tilde{\mathcal{A}}_{k_m}^\alpha\tilde{\mathcal{A}}_{k_l}^\alpha, \tag{9}$$

$$\tilde{\mathcal{A}}_{k_m}^\alpha \equiv \langle\tilde{u}^\alpha|\partial_{k_m}\tilde{u}^\alpha\rangle, \quad |\tilde{u}^\alpha(k)\rangle \equiv |u^\alpha(k,\omega)\rangle_{|\omega=\epsilon_{\alpha,k}} \tag{10}$$

(see Fig. 1). Therefore, it is tempting to regard that the separate definition of the electric and magnetic fields, i.e., Eqs. (2) and (5), is a sort of redundancy. Physically, this is somewhat sensible, since "sharply peaked" quasiparticles ("infinite" lifetime with a definite dispersion relation $\omega=\epsilon_{\alpha,k}$) should be linked with those wave functions which are defined only in this codimensional subspace of the ω-k Euclidean space. Such thinking suggests that only derivatives along this subspace are meaningful, which thereby involve appropriate linear combinations of ω and k derivatives.

Nevertheless, it can be shown that the dual electric and magnetic fields defined in the ω-k Euclidean space do have distinct physical meanings. Specifically, we will show that these two fields enter into the linear response of the spectral *weight* to the applied electric and magnetic fields, respectively [see Eqs. (84) and (88)].[13] This conclusion is obtained by solving the quasiparticle spectral function perturbatively with respect to the gradient expansion. Therein, we observe, in both U(1) and SU(2) cases, that these Berry's curvatures characterize the first order correction to the weight of the quasiparticle spectral function. Since the spectral weight, in principle, can be detected in a *momentum-resolved* way, one

may at least envision several photoemission experiments as candidate tools to make a "contour map" of the dual electric and magnetic fields, separately (see Sec. IV).

A momentum-resolved measurement of the dual electromagnetic fields is of interest not only as a matter of principle but also potentially in diverse experimentally active areas such as anomalous and spin Hall effects in metals and semiconductors and unconventional superconductivity in some ferromagnetic metals. Recent experimental activities in metals[14–16] and semiconductors[17,34] have focused on novel observations of anomalous and spin Hall effects, leading to controversy over the origin of these effects. On the one hand are a variety of proposals of *intrinsic* effects, related to Berry phases and the above-defined dual electromagnetic fields in the clean band theory limit.[6,18] However, such Hall effects could also be of an *extrinsic* origin, i.e., a result of spin-dependent scattering of electrons from the spin-orbit potential of *impurity* atoms.[19] Currently, the discrimination between these scenarios is rather indirect and based primarily on the comparison to diverse theoretical model calculations and approximations whose physical applicability is difficult to judge.[20] If one could somehow *experimentally* "visualize" the distribution of these dual electromagnetic fields in a momentum space, independently from Hall and galvanomagnetic measurements, experimentalists could readily judge for themselves whether intrinsic or extrinsic contributions are dominant in their material sample by comparing with their transport measurements.[35]

A second potential use for a measurement of dual electromagnetic fields stems for a recent proposal by Shi and Niu[21] of a new formulation of the many-body problem based on the noncommutative quantum mechanics. They begin with a quantization of the EOM given in Eq. (1), leading to a nonzero commutator between position operators, $[\hat{R}_j,\hat{R}_m]=i\epsilon_{jml}\mathcal{B}_l(\hat{k})$,[7] in addition to the usual commutators of momenta and position, i.e., $[\hat{k}_j,\hat{k}_m]=i\epsilon_{jml}\mathbf{b}_l(\hat{R})$ and $[\hat{R}_j,\hat{k}_m]=i\delta_{jm}$. As is the case with the momentum in the presence of a real magnetic field $\mathbf{b}$, one can also introduce the *canonical* position operator $\hat{R}'_j\equiv\hat{R}_j-i\mathcal{A}_j(\hat{k})$ such that their commutators become free from the dual magnetic field; $[\hat{R}'_j,\hat{R}'_m]=0$. At a price for this, however, the electron-electron interaction acquires an additional phase factor which can transform a repulsive electron-electron interaction into an attractive one. Observing this, Shi and Niu attributed to the k-space Berry phase a unique origin of the superconductivities realized only within ferromagnetic metals, such as in UGe_2, $ZrZn_2$, and URhGe. A detailed experimental information constraining $\mathcal{B}(k)$ as a function of k, in combination with a measurement of band dispersions, would clearly provide these effective theoretical approaches an *ab initio* model Hamiltonian, only to aid a quantitative comparison with material physics.

The structure of this paper is as follows. In Sec. II, we introduce a quite general multiple-band continuum model and the Keldysh equation for the lesser (greater) Green's function and spectral function. In Sec. III, based on this (dissipationless) Keldysh equation, we provide a systematic procedure for carrying out the projection into the low-energy band, so as to obtain the effective (reduced) Keldysh equation for those Green's functions of quasiparticles forming a Fermi surface.

Thanks to the Fermi liquid assumption, this effective Keldysh equation can be further solved in favor of the spectral function, perturbatively in the gradient expansion. Then, we observe in Sec. IV that the Berry's curvatures, i.e., both dual electric fields and magnetic fields, indeed enter into the first order correction to the renormalization factor for quasiparticles. Using this solution for the spectral function, we further derive in Sec. V the effective Boltzmann equation for the occupation number of quasiparticles in the phase space. Based on this equation, we can finally read off the U(1) effective EOM in interacting Fermi liquids. Section VI is devoted to the summary and discussions of the present paper.

A number of appendices describe other topics useful in understanding the main text in more detail. For completeness, we briefly review the Keldysh formalism and our notations in Appendix A. Appendix B is devoted to demonstrating the logical consistency between our derived U(1) effective EOM and the so-called Ishikawa-Matsuyawa-Haldane formula[22] within the linear response regime. Appendix C describes how a U(1) magnetic monopole "current" in ω-k space (which extends the notion of magnetic monopole density in k space into the ω-k space) determines the distribution of dual electromagnetic fields in the U(1) case. As a specific example which gives a quantitative idea of the significance of the many-body correction, i.e., $\mathcal{E}$, we also present in Appendix D some specific model calculations based on an electron-phonon coupling Hamiltonian.

II. KELDYSH FORMALISM

A. Effective continuum model

We begin with a general semimicroscopic model in which the electronic spectrum is described by a $k\cdot p$-type expansion about some (arbitrary) point in the Brillouin zone. One may keep as many bands as are deemed close enough in energy to be relevant to the physics, and our arguments do not depend on the order in the expansion in k. Familiar examples would be the multiple-band Luttinger models, Dresselhaus models, and Rashba model commonly studied in semiconductors, in which the expansion point is chosen at the Γ point. The advantage of this formulation is that we can Fourier transform, in a usual way, to a continuous real space coordinate r, i.e., $k\rightarrow -i\nabla_r$. The noninteracting Hamiltonian is thereby expressed in terms of the slowly varying "envelope fields" $\psi_\alpha(r)$:

$$\mathcal{H}_0=\sum_{\alpha,\alpha'}\int dr\psi^\dagger_\alpha(r)[\hat{H}_0(-i\nabla_r,r,t)]_{\alpha\alpha'}\psi_{\alpha'}(r), \qquad (11)$$

where the band index $\alpha^{(\prime)}$ runs from 1 to N_b. Specific information about lattice, orbital, and spin-orbit couplings are encoded into the matrix structure of $[\hat{H}_0]$. Following the standard literatures,[23] we will employ a short-ranged electron-electron interaction potential,

$$\mathcal{H}_1 = \sum \int \int V_{\alpha_1\alpha_2\alpha_2'\alpha_1'}(r_1,r_2)\psi^\dagger_{\alpha_1}(r_1)\psi^\dagger_{\alpha_2}(r_2)\psi_{\alpha_2'}(r_2)\psi_{\alpha_1'}(r_1), \tag{12}$$

though the form of our results does not depend in detail on this.

B. Dissipationless Keldysh equation

Apart from the perturbation theory at equilibrium, the Keldysh formalism is constituted by the lesser and greater Green's functions,

$$\mathsf{g}^{>}(1|1') \equiv -i\langle\psi(1)\psi^\dagger(1')\rangle,$$

$$\mathsf{g}^{<}(1|1') \equiv i\langle\psi^\dagger(1')\psi(1)\rangle,$$

where the time evolution of $\psi(1) \equiv \psi_{\alpha_1}(r_1,t_1)$ is determined by the interacting Hamiltonian introduced above,

$$-i\frac{\partial\psi(1)}{\partial t_1} \equiv [\mathcal{H}_0 + \mathcal{H}_1, \psi(1)].$$

Because these Green's functions are *not* the time-ordered ones, they acquire the translational symmetry in space and *time* at equilibrium. Furthermore, they are always Hermite matrices with respect to space and time coordinates and band indices.

Putting its derivation aside (see Appendix A), let us begin with the Keldysh equation for these lesser and greater Green's functions,

$$[G_0^{-1} - \Sigma^{\mathrm{HF}} - \sigma, \mathsf{g}^{<(>)}]_{\otimes,-} - [\Sigma_c^{<(>)}, \mathsf{b}]_{\otimes,-} = \frac{1}{2}[\Sigma_c^{>}, \mathsf{g}^{<}]_{\otimes,+} - \frac{1}{2}[\Sigma_c^{<}, \mathsf{g}^{>}]_{\otimes,+}. \tag{13}$$

The (anti)commutator here is defined by the convolution with respect to time t, space r, and band index α,

$$[\hat{B},\hat{C}]_{\otimes,\pm}(1,1') \equiv \int d\bar{1}\hat{B}(1|\bar{1})\cdot\hat{C}(\bar{1}|1') \pm \int d\bar{1}\hat{C}(1|\bar{1})\cdot\hat{B}(\bar{1}|1'),$$

with $\int d1 \equiv \Sigma_{\alpha_1}\int dr_1 dt_1$. We denoted the bare Green's function as $\hat{G}_0$, which is composed of the quadratic part of the Hamiltonian,

$$\hat{G}_0^{-1}(1|1') \equiv [i\partial_{t_1}\hat{1} - \hat{H}_0(-i\nabla_{r_1}, r_1, t_1)]_{\alpha_1\alpha_1'}\delta(t_1 - t_1')\delta^d(r_1 - r_1').$$

d above represents the spatial dimension of our system.

Electron-electron interaction effects, on the other hand, are encoded in the self-energy such as $\hat{\Sigma}^{\mathrm{HF}}(1|1')$ and $\hat{\Sigma}_c^{<(>)}(1|1')$, which are self-consistently given in terms of lesser and greater Green's functions [see Eqs. (A5)–(A7) in Appendix A]. The former self-energy is the Hartree-Fock part, which is temporally instantaneous, i.e., $\Sigma^{\mathrm{HF}}(1|1') \sim \delta(t_1 - t_{1'})$. The latter one is usually dubbed as the lesser (greater) collisional self-energy, which is at least second order in electron-electron interactions and thus temporally noninstantaneous. The physical role of this collisional self-energy is twofold, corresponding to its following decomposition,

$$\sigma(1|1') \equiv \frac{1}{2}\frac{t_1 - t_{1'}}{|t_1 - t_{1'}|}(\Sigma_c^{>}(1|1') - \Sigma_c^{<}(1|1')) \equiv \frac{1}{2}(\Sigma^R(1|1') + \Sigma^A(1|1')), \tag{14}$$

$$\Gamma(1|1') \equiv i(\Sigma_c^{>}(1|1') - \Sigma_c^{<}(1|1')) \equiv i(\Sigma^R(1|1') - \Sigma^A(1|1')), \tag{15}$$

with $\sigma(1|1')^* \equiv \sigma(1'|1)$ and $\Gamma(1|1')^* \equiv \Gamma(1'|1)$. $\Sigma^R(1|1')$ and $\Sigma^A(1|1')$ on the right hand side denote the retarded and advanced self-energies, respectively. Thus, $\sigma(1|1')$ stands for the real (Hermitian) part of the self-energy associated with the time-ordered Green's function, which plays the role of the renormalization of the quasiparticle energy and wave function. On the other hand, $\Gamma(1|1')$ is its imaginary (anti-Hermitian) part, bringing about a finite lifetime for quasiparticles.

In a same way, we can define the real and imaginary part of the Green's function;

$$\mathsf{b}(1|1') \equiv \frac{1}{2}\frac{t_1 - t_{1'}}{|t_1 - t_{1'}|}(\mathsf{g}^{>}(1|1') - \mathsf{g}^{<}(1|1')) \equiv \frac{1}{2}(\mathsf{g}^R(1|1') + \mathsf{g}^A(1|1')), \tag{16}$$

$$\mathsf{A}(1|1') \equiv i(\mathsf{g}^{>}(1|1') - \mathsf{g}^{<}(1|1')) \equiv i(\mathsf{g}^R(1|1') - \mathsf{g}^A(1|1')), \tag{17}$$

with $\mathsf{b}(1|1')^* \equiv \mathsf{b}(1'|1)$ and $\mathsf{A}(1|1')^* \equiv \mathsf{A}(1'|1)$. Note that especially the latter one is nothing but the spectral function. As is clear from Eq. (13), the Keldysh equation for this spectral function is composed only by the real and imaginary part of the self-energy and Green's functions introduced above,

$$[G_0^{-1} - \Sigma^{\mathrm{HF}} - \sigma, \mathsf{A}]_{\otimes,-} - [\Gamma, \mathsf{b}]_{\otimes,-} = 0. \tag{18}$$

We will dub $G_0^{-1}(1|1') - \Sigma^{\mathrm{HF}}(1|1') - \sigma(1|1')$ as a "Lagrangian" $\mathsf{L}(1|1')$ in a sense that, when Fourier transformed with respect to its relative coordinate, i.e., $1-1'$, it reduces to ω minus a renormalized Hamiltonian for quasiparticles (qp). Namely, its eigenvalues specify the renormalized energy dispersions for qp as their zeros with respect to ω, while their corresponding eigenvectors constitute an orthogonal set. The latter of which can therefore be regarded as (a periodic part of) the renormalized Bloch wave function. Accordingly, we are led to define the Berry curvatures and gauge connections in the dual space in terms of the unitary matrix diagonalizing this Lagrangian [see Eqs. (39), (38), and (21), respectively]. As is shown in this paper, our effective Boltzmann and Keldysh equation legally derived via the projection process claims that the Berry's curvatures thus introduced indeed govern the effective EOM for qp in interacting Fermi systems.

On the other hand, the commutator between the anti-Hermitian part of the self-energy Γ and the Hermitian part of the Green's function b introduces a finite *lifetime* of qp, i.e., the broadening of the spectral functions. When it comes to the qp closed to a Fermi surface, however, the Fourier-transformed Γ as a function of ω becomes as small as $O((\omega-\mu)^2, T^2)$. Namely, the second expression of Eq. (15) dictates that, when analytically continued from a Matsubara Green's function, Γ at equilibrium is composed of a delta function, which imposes the energy conservation on its internal lines. As a result, the momentum integral regions associated with the internal lines are restricted to be only near a Fermi surface at low temperature.[24]

Γ in those electron-boson coupled systems with bosons having a finite excitation energy gets even smaller than this power-law decay. When $|\omega-\mu|$ is much smaller than this excitation energy ω_0, (the lowest order) perturbative calculations readily show that $\hat{\Gamma}$ vanishes exponentially, such as $e^{-|\omega-\mu|/\omega_0}$ or e^{-T/ω_0} (see Appendix D). Anyway, in both of these two cases, the broadening of the qp spectral function at sufficiently low T is by far smaller than the thermal line broadening of the spectral function ($\sim T$), which validates the so-called adiabatic assumption of the Fermi liquid theory.

Meanwhile, the Hermitian part of the collisional self-energy $\hat{\sigma}$ remains finite even *on* a Fermi surface ($\omega=\mu$) at *zero* temperature ($T=0$). Namely, Eq. (14) indicates that, when analytically continued from a Matsubara Green's function, $\hat{\sigma}$ at equilibrium is composed of the principal integral rather than the delta function and thus free from the energy conservation imposed on its internal lines (see Appendix D for some example). As such, apart from the lifetime part Γ, its momentum integral region for the internal lines is not restricted near the Fermi surface, which even causes the ultraviolet cutoff dependence of $\hat{\sigma}$ at $T=0$.

Because of these two different features generic in $\hat{\Gamma}$ and $\hat{\sigma}$, we ignore in this paper the (intrinsic) lifetime effect $\hat{\Gamma}$ while fully take into account the renormalization effects due to $\hat{\sigma}$. Namely, instead of Eq. (18), we begin with the following *dissipationless* Keldysh equation for spectral functions:

$$[\hat{\mathsf{L}},\hat{\mathsf{A}}]_{\otimes,-}=\hat{0}. \tag{19}$$

Even this SU(N_b) dissipationless Keldysh equation is still nontrivial due to the presence of the *band index* whose effect is the central issue of this paper. In the following, we will present a general method of projecting out irrelevant band degrees of freedom associated with fully occupied bands and fully empty bands, so as to derive perturbatively the reduced Keldysh (kinetic) equation only for the relevant bands constituting a Fermi surface.

III. REDUCED KELDYSH EQUATION

A. Gradient expansion

We will derive these reduced Keldysh equations perturbatively with respect to the *gradient expansion*. The coupling constant of this expansion is a dimensionless quantity which measures how much a system is disequilibrated. To define this expansion accurately, notice first that the lesser and greater Green's functions *at equilibrium* acquire the translational invariance in space and time coordinates; $\mathsf{g}^{<(>)}(1|1')=\mathsf{g}^{<(>)}_{\alpha_1\alpha_{1'}}(r_1-r_{1'},t_1-t_{1'})$. Being given in terms of these Green's functions, the lesser and greater self-energy and Lagrangian also become translationally invariant at equilibrium; $\mathsf{L}(1|1')=\mathsf{L}_{\alpha_1\alpha_{1'}}(r_1-r_{1'},t_1-t_{1'})$. As such, when Fourier transformed, the convolution encoded in the dissipationless Keldysh equation reduces a simple product only with respect to band index,

$$[\hat{\mathsf{L}}(q,\omega),\hat{\mathsf{A}}(q,\omega)]_-=\hat{0},$$

$$\begin{bmatrix}\mathsf{L}_{\alpha\beta}\\ \mathsf{A}_{\alpha\beta}\end{bmatrix}(q,\omega)\equiv\int drdte^{-iqr+i\omega t}\begin{bmatrix}\mathsf{L}_{\alpha\beta}\\ \mathsf{A}_{\alpha\beta}\end{bmatrix}(r,t).$$

Accordingly, we have only to diagonalize the Lagrangian so that an arbitrary diagonal $\hat{\mathsf{A}}$ in this eigenbasis satisfies Eq. (19) at equilibrium. We will regard this trivial limit as the nonperturbed case and take into account disequilibrations as perturbations.

When a system is disequilibrated, the lesser and greater Green's functions generally depend on the *center of mass coordinate* in space and time, i.e., $R\equiv\frac{r_1+r_{1'}}{2}$ and $T\equiv\frac{t_1+t_{1'}}{2}$;

$$\mathsf{g}^{<(>)}(1|1')=\mathsf{g}^{<(>)}_{\alpha_1\alpha_{1'}}(r,t;R,T).$$

However, as long as a system is not so far from its equilibrium case, their dependences on R and T are slowly varying in comparison with the lattice spacing and inverse of bandwidth, respectively. As such, we could quantify the "distance" from equilibrium by a dimensionless ratio between this slowly varying length (time) scale and a lattice spacing (inverse of the bandwidth). To extract the latter length and/or time scale, we have only to Fourier transform the relative coordinate, i.e., r and t, so that the crystal momentum q and energy (frequency) ω are introduced,[25]

$$\hat{\mathsf{g}}^{<}(\omega,q;T,R)=-i\int drdte^{-iqr+i\omega t}\hat{\mathsf{g}}^{<}(r,t;R,T),$$

$$\hat{\mathsf{g}}^{>}(\omega,q;T,R)=i\int drdte^{-iqr+i\omega t}\hat{\mathsf{g}}^{>}(r,t;R,T).$$

Then, the derivatives of these Green's functions with respect to q and ω are quantities of the order of the Fermi length and inverse of a bandwidth, respectively. Thus, the (inner) products between the derivatives of these Green's functions with respect to $Q\equiv(\omega,q)$ and those with respect to $X\equiv(T,R)$ can be regarded as a small dimensionless quantity, as far as a system is only weakly disequilibrated,

$$\hat{\mathsf{g}}^a\cdot\hat{\mathsf{g}}^b\gg\partial_X\hat{\mathsf{g}}^a\cdot\partial_Q\hat{\mathsf{g}}^b\gg\partial_X\partial_{X'}\hat{\mathsf{g}}^a\cdot\partial_Q\partial_{Q'}\hat{\mathsf{g}}^b,\ldots.$$

The superscripts a and b specify the lesser or greater Green's functions. Since the Lagrangian and spectral function are given in terms of these functions self-consistently, we can readily adopt the following relations also:

$$\hat{\mathsf{L}}\cdot\hat{\mathsf{A}}\gg\partial_X\hat{\mathsf{L}}\cdot\partial_Q\hat{\mathsf{A}},\partial_Q\hat{\mathsf{L}}\cdot\partial_X\hat{\mathsf{A}}\gg\partial_X\partial_{X'}\hat{\mathsf{L}}\cdot\partial_Q\partial_{Q'}\hat{\mathsf{A}},\ldots.$$

Observing this, we expand the convolution in the (dissipationless) Keldysh equation, in powers of $\partial_Q \partial_X \equiv -\partial_\omega \partial_T + \partial_{q_j} \partial_{R_j}$,

$$-[\hat{\mathsf{L}},\hat{\mathsf{A}}]_- = \frac{i}{2}[\partial_{X_j}\hat{\mathsf{L}},\partial_{Q_j}\hat{\mathsf{A}}]_+ - \frac{1}{8}([\partial_{X_j}\partial_{X_k}\hat{\mathsf{L}},\partial_{Q_j}\partial_{Q_k}\hat{\mathsf{A}}]_- - \{X_k \leftrightarrow Q_k\}) - \{X_j \leftrightarrow Q_j\} + \cdots . \quad (20)$$

While we kept up to the second order, one could explicitly write down the higher than this by using the following formula for the Moyal product,[26]

$$(A \otimes B)(Q;X) \equiv \int d(1-1')e^{iQ\cdot(1-1')} \int d\bar{1} A(1|\bar{1})B(\bar{1}|1') = e^{i(1/2)(\partial_X^A \partial_Q^B - \partial_Q^A \partial_X^B)} A(Q;X)B(Q;X),$$

where $\partial_{X_i}^A \partial_{Q_j}^B \partial_{Q_m}^A \partial_{X_l}^B (AB) \equiv (\partial_{X_i}\partial_{Q_m} A)(\partial_{Q_j}\partial_{X_l} B)$. Note that the j and k summations in Eq. (20) run from 0 to d, which will be made implicit from now on. The (anti)commutators in Eq. (20) are taken only with respect to band indices, while the nonlocal correlation effect encoded into the space-time convolution of Eq. (19) is now perturbatively taken into account via the gradient expansion.

B. Projection process

Our projection process is nothing but to solve Eq. (20), perturbatively in the gradient expansion, using the *substitution* method. To be more specific, we are looking for $\hat{\mathsf{A}}$ which satisfies this equation at a given order accuracy in the gradient expansion. To do this in a well-controlled fashion, we will first solve the off-diagonal elements of the spectral function in favor of its diagonal elements by looking into the off-diagonal components of the matrix-formed Keldysh equation (KE) given in Eq. (20). Substituting these solutions back into the diagonal components of the same KE, we therefore obtain sort of differential equations given only for the diagonal elements of the spectral function (Secs. III B–III D). Then, we will further determine appropriate form of these diagonal elements, such that these differential equations are satisfied (Sec. IV).

As in standard perturbation theories, we begin with diagonalizing the zeroth order part. Introduce the unitary matrix which diagonalizes the Lagrangian $\hat{\mathsf{L}}$ on the left hand side of Eq. (20),

$$\hat{L}_d \equiv \hat{U}^\dagger \hat{\mathsf{L}} \hat{U}. \quad (21)$$

Then, a spectral function in this basis has only to be diagonal so as to satisfy Eq. (20) at the zeroth order,

$$\hat{A} \equiv \hat{U}^\dagger \hat{\mathsf{A}} \hat{U} = \begin{bmatrix} A_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_{N_b} \end{bmatrix}. \quad (22)$$

We used sans serif for the spectral and Green's function represented in the old basis, while roman for those in the new basis.

While each diagonal element can be arbitrary at this level, from the physical point of view, they should be delta functions,

$$A_\alpha \equiv \delta(L_{d,\alpha}^{(0)}) = Z_\alpha^{(0)} \delta(\omega - \epsilon_{\alpha,q}^{(0)}). \quad (23)$$

Namely, $\epsilon_{\alpha,q}$ above denotes the (renormalized) energy dispersion for the αth band qp, while Z_α stands for this qp spectral weight. The superscript (0) simply represents that they are quantities of the zeroth order in gradient expansion. When employing this form as the zeroth order solution, we will actually be able to satisfy the Keldysh equation up to the first order in the gradient expansion [see Eq. (74)]. Thus, we can justify *a posteriori* that Eq. (23) is the appropriate zeroth order solution, based on which its higher order correction can be built up (see also the arguments in Sec. IV).

Under this unitary transformation, the usual derivative encoded in Eq. (20) is replaced by the "covariant" derivative,

$$\hat{U}^\dagger(\partial_X \hat{\mathsf{B}})\hat{U} = [\hat{D}_X, \hat{B}] \equiv \partial_X \hat{B} + \hat{\mathcal{A}}_X \hat{B} - \hat{B}\hat{\mathcal{A}}_X, \quad (24)$$

with $\hat{B} = U^\dagger \cdot \hat{\mathsf{B}} \cdot U$ and $\hat{\mathcal{A}}_X = \hat{U}^\dagger \partial_X \hat{U}$. Namely, in terms of this derivative, our matrix-formed differential equation reads

$$-[\hat{L}_d, \hat{A}] = \frac{i}{2}[[\hat{D}_{X_j}, \hat{L}_d], [\hat{D}_{Q_j}\hat{A}]]_+ + \cdots \equiv \hat{F}(\hat{A}). \quad (25)$$

To find the spectral function satisfying Eq. (25) up to higher order in the gradient expansion, let us next look into the off-diagonal components of this covariant differential equation,

$$-L_{d,\alpha}A_{\alpha\beta} + A_{\alpha\beta}L_{d,\beta} = F_{\alpha\beta}(\{A_\gamma\},\{A_{\gamma\eta}\}). \quad (26)$$

$F_{\alpha\beta}$ is a functional of a set of diagonal elements of the spectral function, i.e., $\{A_\gamma \equiv A_{\gamma\gamma}\}$, and a set of its off-diagonal elements $\{A_{\gamma\eta}\}$ $(\gamma \neq \eta)$. Notice first that the right hand side of Eq. (26) is at least first order in gradient expansion. On one hand, its left hand side is basically proportional to a direct band gap between αth band and βth band, which is finite even at equilibrium, i.e., $L_{d,\alpha} \neq L_{d,\beta}$. Thus, the off-diagonal elements of the spectral function are of the order of $\mathcal{O}(|\partial_X \partial_Q|)$, while its diagonal elements remain finite even at equilibrium.

As such, we first solve these off-diagonal elements in favor of the diagonal elements of the spectral function, iteratively in gradient expansion. Specifically, to the first order's accuracy, we have only to replace the off-diagonal elements in the right hand side of Eq. (26) by zero,

$$A_{\alpha\beta} = F_{\alpha\beta}(\{A_\gamma\},\{0\}) \cdot (L_{d,\beta} - L_{d,\alpha})^{-1} \equiv A_{\alpha\beta}^{(1)}. \quad (27)$$

Using this, one could further obtain the solution of $A_{\alpha\beta}$ to the second order accuracy,

$$A_{\alpha\beta}^{(2)}(\{A_\gamma\}) = F_{\alpha\beta}(\{A_\gamma\},\{A_{\gamma\eta}^{(1)}\}), \quad (28)$$

or higher than that,

$$A_{\alpha\beta}^{(n+1)}(\{A_\gamma\}) = F_{\alpha\beta}(\{A_\gamma\},\{A_{\gamma\eta}^{(n)}\}). \quad (29)$$

When n being infinity, this clearly becomes an exact relation between diagonal elements and off-diagonal elements.

Substituting these solutions into the diagonal components

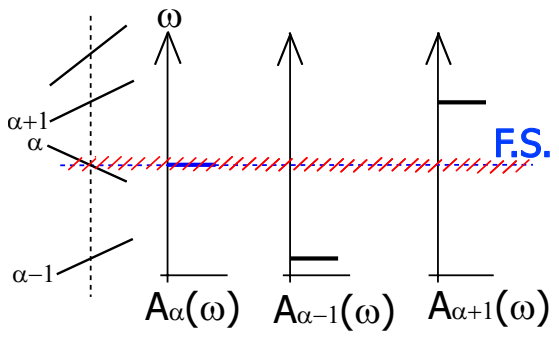

FIG. 2. (Color online) A schematic picture of the spectral function for the αth band and its neighboring bands. The dotted (blue) horizontal line stands for a Fermi surface, while the hatched (red) area specifies the frequency region we are interested in.

of the covariant differential equation, we then have N_b *decoupled* equations which are given in terms only of a set of diagonal elements of the spectral function,

$$0 = F_\alpha(\{A_\gamma\},\{A_{\gamma\eta}\}) = F_\alpha(\{A_\gamma\},\{A_{\gamma\eta} \equiv A^{(\infty)}_{\gamma\eta}(\{A_\gamma\})\}), \tag{30}$$

with $F_\alpha \equiv F_{\alpha\alpha}$. Then, a remaining task is to determine a set of N_b diagonal elements of $\hat{A}$, such that they observe these equations. $\{A_\gamma\}$ thus obtained in combination with $A_{\gamma\eta} \equiv A^{(\infty)}_{\gamma\eta}(\{A_\gamma\})$ are, in principle, equivalent to the exact solution of the an original dissipationless Keldysh equation, i.e., Eq. (20).

However, the exact solution is apparently impossible, since it would require us, for example, to perform the iteration of Eq. (29) at an infinite time. Thereby, we are going to indulge ourselves in executing this sequence of the process, *up to a given order* in gradient expansion. The highest order up to which we have succeeded in obtaining Eq. (30) is currently the second order. Up to this order, we can readily ignore the third order gradient expansion term (and higher than that) denoted by "$\cdots$" in Eq. (20). Furthermore, $F_{\alpha\alpha}$ already containing at least one pair of $\partial_X \partial_Q$, the difference between Eqs. (27) and (28) ends up with the third order contributions when substituted into $F_{\alpha\alpha}$ as in Eq. (30). Thus, we are ready to use Eq. (27).

The final simplification we will employ is that, for given k, R, and T, each quasiparticle band has spectral weights at energetically well separated regions from one another. To be concrete, let us refer to a band which contains a Fermi surface as the αth band. Then, the diagonal elements of the spectral function corresponding to the other bands, i.e., $A_{\gamma\neq\alpha}$, have negligible weights at the low-energy region, i.e., $|\omega - \epsilon_\alpha| \simeq |\omega - \mu| \ll \min_\beta |\Delta_{\alpha\beta}|$. This can be seen precisely at equilibrium, where $A_{\gamma\neq\alpha}$ is sharply peaked at an energy region separated from μ by the direct band gap $\Delta_{\gamma\alpha}$ (see Fig. 2). Even off equilibrium, higher order gradient expansion corrections to A_γ turn out not to make additional *incoherent* weights other than the delta function we originally have at equilibrium. Namely, the corrections to A_γ appear in Eq. (23) only as the energy dispersion *shift* and the *additive* spectral weight [see, for example, Eq. (88)],

$$\epsilon^{(0)}_\alpha \to \epsilon^{(0)}_\alpha + \epsilon^{(1)}_\alpha, \quad Z^{(0)}_\alpha \to Z^{(0)}_\alpha + Z^{(1)}_\alpha.$$

Accordingly, as far as this energy shift, i.e., $\epsilon^{(1)}_\gamma$, does not change the relative position of each quasiparticle energy dispersions, we are *a posteriori* allowed to replace $A_{\gamma\neq\alpha}$ by zero, for $|\omega - \epsilon_\alpha| \ll \min_\beta \Delta_{\alpha\beta}$.

To summarize the simplifications possible at the second order analysis, we have only to derive the following equations:

$$0 = F_\alpha(\{A_\alpha, A_{\gamma\neq\alpha} \equiv 0\},\{A^{(1)}_{\gamma\eta}\}),$$

$$A^{(1)}_{\gamma\eta} \equiv F_{\gamma\eta}(\{A_\alpha, A_{\gamma\neq\alpha} \equiv 0\},\{0\}) \cdot (L_{d,\eta} - L_{d,\gamma})^{-1}. \tag{31}$$

In Sec. III D we will substitute the latter into the former, so as to obtain the (differential) equation only for A_α. We will dub the equation thus obtained as a *reduced Keldysh equation* (RKE). In Sec. IV, we will further find a A_α satisfying this reduced Keldysh equation. Thereby, we actually observe that A_α thus obtained is sharply peaked at $\omega = \epsilon_\alpha$ while having no incoherent weights at high-energy sides. This observation will support *a posteriori* the logical consistency built in our prescription described above.

C. SU(2) Fermi liquids and additional coupling constant

In the argument of Sec. III B, we have implicitly assumed that a Fermi surface in question is composed only by a single band [especially Eqs. (21) and (22)]. In general, this is true either in those metals without any centrosymmetric lattice point or in ferromagnetic metals. In such FLs, there remains only a charge degree of freedom, while the (pseudo)spin degree of freedom at each k point is usually *quenched*. This charge degree of freedom is then described by the spectral function A_α, which is a *scalar* quantity. Accordingly, the RKE, i.e., Eq. (31), becomes just a differential equation for this scalar function, which we can name as a U(1) RKE.

On one hand, in usual paramagnetic metals having a centrosymmetric lattice point, each k point is (at least) doubly degenerate, describing spin degree of freedom for quasiparticles. Namely, eigenvalues of our Lagrangian are always twofold at equilibrium. Thus, a Fermi surface is composed by two degenerate bands, which we could name as SU(2) FL,

$$\hat{L}_d = \hat{U}^\dagger \hat{\mathsf{L}} \hat{U} = \begin{bmatrix} \hat{L}_{d,1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \hat{L}_{d,N_b} \end{bmatrix}.$$

Here, 2×2 matrices $\hat{L}_{d,\gamma}$ should be proportional to a unit matrix at equilibrium.

As such, the spectral function which satisfies Eq. (20) at the zeroth order in gradient expansion has only to be block diagonalized,

$$\hat{A} = \hat{U}^\dagger \hat{\mathsf{A}} \hat{U} = \begin{bmatrix} \hat{A}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \hat{A}_{N_b} \end{bmatrix}, \tag{32}$$

with arbitrary 2×2 matrices $\hat{A}_\gamma$ ($\gamma = 1, \ldots, N_b$).

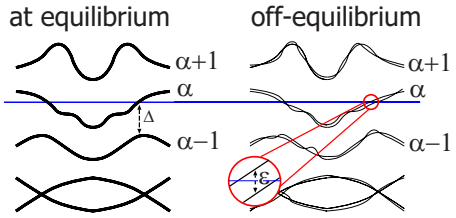

FIG. 3. (Color online) A schematic picture of the energy dispersion in SU(2) FLs. (Right) Each dispersion is doubly degenerate at equilibrium. (Left) When a system is weakly disequilibrated, the double degeneracy at each k point is lifted. The associated splitting energy, i.e., ϵ, is, however, much smaller than the typical band gap and/or width at equilibrium, i.e., Δ.

Corresponding to this generic degeneracy at the zeroth order, we will derive the RKEs in SU(2) FLs in favor of these 2×2 spectral functions, i.e., $\hat{A}_\gamma$ ($\gamma=1,\ldots,N_b$), so that spin and charge degrees of freedom are treated on an equal footing. As will be shown later, the RKE thus derived becomes a 2×2 matrix-formed differential equation, which we will dub as SU(2) RKE.

Apart from relatively minor modifications, the derivation of the SU(2) RKE also goes along with the same procedure as in U(1) case. Specifically, it also begins with the interband components of KE, which now take a 2×2 matrix form,

$$-\hat{L}_{d,\gamma}\hat{A}_{\gamma\eta}+\hat{A}_{\gamma\eta}\hat{L}_{d,\eta}=\hat{F}_{\gamma\eta}(\{\hat{A}_\gamma\},\{\hat{A}_{\gamma\eta}\}), \tag{33}$$

where $\hat{A}_{\gamma\eta}$ is a 2×2 matrix, connecting the γth band and ηth band.

Being proportional to a unit matrix at equilibrium, 2×2 matrices $\hat{L}_{d,\gamma}$ may be decoupled into its zeroth order part and a small degeneracy lifting part,

$$\hat{L}_{d,\gamma}=L^{(0)}_{d,\gamma}\hat{1}-\epsilon_\gamma\hat{\sigma}_z. \tag{34}$$

A finite $\hat{\epsilon}_\gamma\equiv\epsilon_\gamma\hat{\sigma_z}$ is generally originated from weak R and T dependences of the Green's functions (see Fig. 3). Thus, ϵ_γ (divided by a characteristic bandwidth) should be treated as a same order of quantity as $|\partial_X\partial_Q|$. To convince ourselves of this more directly, consider a specific situation in the presence of a small magnetic field $\mathbf{b}$, or equivalently a slowly varying magnetic gauge potential. In such a case, the Zeeman coupling energy between (bare) spin and $\mathbf{b}$ clearly should be included in $\hat{\epsilon}_\gamma$. On one hand, $|\partial_X\partial_Q|$ contribution turns out to be proportional to a spatial derivative of the external gauge field, which is therefore proportional to $\mathbf{b}$ also [for example, compare Eq. (48) with Eqs. (70) and (71)]. As is obvious from this example, $|\partial_X\partial_Q|\equiv\lambda_1$ and $\epsilon/\Delta\equiv\lambda_2$ should be treated as same order of quantities,

$$\lambda_1\sim\lambda_2\sim\lambda.$$

Observing the interband components of the covariant differential equations, i.e., Eq. (33), we first relate the interband elements of $\hat{A}$ with a set of N_b intraband elements of $\hat{A}$. To the first order in λ_1 or λ_2, i.e., $\mathcal{O}(\lambda_1,\lambda_2)$, we have

$$\hat{A}_{\gamma\eta}=\hat{F}_{\gamma\eta}(\{\hat{A}_\gamma\},\{\hat{0}\})\cdot(\hat{L}_{d,\eta}-L^{(0)}_{d,\gamma}\hat{1})^{-1}\equiv\hat{A}^{(1)}_{\gamma\eta}.$$

Substituting these first order solutions into the *intraband* components of KE, we then obtain the following SU(2) RKE to the accuracy of $\mathcal{O}(\lambda_1^2,\lambda_2^2,\lambda_1\lambda_2)$,

$$-[\hat{L}_{d,\alpha},\hat{A}_\alpha]=[\hat{\epsilon}_\alpha,\hat{A}_\alpha]=\hat{F}_\alpha(\{\hat{A}_\alpha,\hat{A}_{\gamma\neq\alpha}\equiv 0\},\{\hat{A}^{(1)}_{\gamma\eta}\}), \tag{35}$$

$$\hat{A}^{(1)}_{\gamma\eta}(\hat{A}_\alpha)\equiv\hat{F}_{\gamma\eta}(\{\hat{A}_\alpha,\hat{A}_{\gamma\neq\alpha}\equiv 0\},\{\hat{A}_{\gamma\eta}\equiv\hat{0}\})\cdot(\hat{L}_{d,\eta}-L^{(0)}_{d,\gamma}\hat{1})^{-1}. \tag{36}$$

The intraband elements of $\hat{A}$ for the bands other than the αth band, i.e., $\hat{A}_{\gamma\neq\alpha}$, were already replaced by zero, because of the same reason as we argued in the U(1) case.

In the next section, we will calculate Eq. (35) in combination with Eq. (36) more explicitly, so as to obtain an actual form of the SU(2) RKE up to the order of $\mathcal{O}(\lambda^2)$. Out of the SU(2) RKE thus derived, one can immediately obtain the RKE in U(1) FLs by regarding $\hat{A}_\alpha$ as a scalar function and putting $\hat{\epsilon}_\alpha$ to be zero.

D. Actual derivations of SU(2) reduced Keldysh equations

In this section, we will perform the actual calculation of Eqs. (35) and (36) in a covariant way, only to arrive at the SU(2) RKE associated with the αth band in question. By referring "in a covariant way" we mean that every step in the following manipulation does not change its explicit form either under an SU(2) rotation $\hat{v}_\alpha$ within the αth band or under an SU($2N_b-2$) rotation $\hat{V}_{\bar{\alpha}}$ within its complementary space;

$$\hat{U}\to\hat{U}\cdot\begin{bmatrix}\hat{v}_\alpha & \hat{0}\\ \hat{0} & \hat{V}_{\bar{\alpha}}\end{bmatrix}. \tag{37}$$

This is because any steps of the actual calculation are composed either by the *interband covariant derivatives* or by the *intraband covariant derivatives*, both of which will be accurately defined in Sec. III D 1. In addition to their definitions, some formula frequently used in Sec. III D 2 will be also summarized in advance for clarity.

1. Arithmetic preliminaries

We have already defined the SU($2N_b$) gauge field and associated "covariant" derivative in Eq. (24). In an analogous way, the SU(2) gauge fields and covariant derivatives for the twofold degenerate αth band are defined as follows:

$$[\hat{D}^\alpha_X,\hat{B}_\alpha]\equiv\partial_X\hat{B}_\alpha+[\hat{\mathcal{A}}^\alpha_X,\hat{B}_\alpha],$$

$$[\hat{\mathcal{A}}^\alpha_X]_{(\sigma|\sigma')}\equiv[\hat{U}^\dagger\partial_X\hat{U}]_{(\alpha\sigma|\alpha\sigma')}, \tag{38}$$

where X could be any coordinates in the phase space. We sub(super)scribe α or $\bar{\alpha}$ such as $\hat{B}_\alpha$ or $\hat{B}_{\bar{\alpha}}$, only to suggest that this matrix is a 2×2 matrix associated with the αth band or a $(2N_b-2)\times(2N_b-2)$ matrix in its complementary space, respectively. When sub(super)scribed "$\alpha\bar{\alpha}$" such as $\hat{B}^{(\alpha\bar{\alpha})}_{\alpha\bar{\alpha}}$, a matrix ought to be regarded as a $2\times(2N_b-2)$ matrix.

Apart from the derivative in the *full* Hilbert space, the

covariant derivatives in its *subspace* do not commute with one another in general,

$$[\hat{D}_X^\alpha,[\hat{D}_{X'}^\alpha,\hat{B}_\alpha]]-[\hat{D}_{X'}^\alpha,[\hat{D}_X^\alpha,\hat{B}_\alpha]]=-i[\hat{\Omega}_{XX'}^\alpha,\hat{B}_\alpha],$$

$$\hat{\Omega}_{XX'}^\alpha \equiv i[\hat{D}_X^\alpha,\hat{D}_{X'}^\alpha]=i\partial_X\hat{\mathcal{A}}_{X'}^\alpha-i\partial_{X'}\hat{\mathcal{A}}_X^\alpha+i[\hat{\mathcal{A}}_X^\alpha,\hat{\mathcal{A}}_{X'}^\alpha]. \tag{39}$$

As is clear from the latter expression, $\hat{\Omega}_{XX'}^\alpha$ would be identical to zero, if this doubly degenerate αth band were to subtend a complete set, i.e., $\Sigma_{\sigma=\pm}|u^{\alpha\sigma}\rangle\langle u^{\alpha\sigma}|\equiv\hat{1}$.

Note that this derivative and the associated curvature $\hat{\Omega}_{XX'}^\alpha$ transform in a covariant way under the unitary transformation in Eq. (37),

$$\hat{B}_\alpha\to\hat{v}_\alpha^\dagger\hat{B}_\alpha\hat{v}_\alpha,\quad [\hat{D}_X^\alpha,\hat{B}_\alpha]\to\hat{v}_\alpha^\dagger[\hat{D}_X^\alpha,\hat{B}_\alpha]\hat{v}_\alpha,$$

$$\hat{\Omega}_{XX'}^\alpha\to\hat{v}_\alpha^\dagger\hat{\Omega}_{XX'}^\alpha\hat{v}_\alpha, \tag{40}$$

while the gauge field $\hat{\mathcal{A}}_X^\alpha$ is not,

$$\hat{\mathcal{A}}_X^\alpha\to\hat{v}_\alpha^\dagger\hat{\mathcal{A}}_X^\alpha\hat{v}_\alpha+\hat{v}_\alpha^\dagger\partial_X\hat{v}_\alpha.$$

In addition to this gauge field in the αth band, we can also define the *interband* gauge fields as the $2\times(2N_b-2)$ off-diagonal blocks of $\hat{\mathcal{A}}_X$,

$$\hat{\mathcal{A}}_X\equiv\hat{U}^\dagger\partial_X\hat{U}\equiv\begin{bmatrix}\hat{\mathcal{A}}_X^\alpha & \hat{\mathcal{A}}_X^{\alpha\bar{\alpha}}\\ \hat{\mathcal{A}}_X^{\bar{\alpha}\alpha} & \hat{\mathcal{A}}_X^{\bar{\alpha}}\end{bmatrix}.$$

Under the unitary transformation defined in Eq. (37), the interband gauge field clearly transforms in a covariant way,

$$\hat{\mathcal{A}}_X^{\alpha\bar{\alpha}}\to\hat{v}_\alpha^\dagger\hat{\mathcal{A}}_X^{\alpha\bar{\alpha}}\hat{V}_{\bar{\alpha}},$$

while the intraband gauge fields such as $\hat{\mathcal{A}}_X^\alpha$ and $\hat{\mathcal{A}}_X^{\bar{\alpha}}$ are not.

Using the latter ones, we can further define a $2\times(2N_b-2)$ matrix-form derivative, which plays role of a sort of interband covariant derivative,

$$[\hat{D}_X^{\alpha\bar{\alpha}},\hat{B}_{\alpha\bar{\alpha}}]\equiv\partial_X\hat{B}_{\alpha\bar{\alpha}}+\hat{\mathcal{A}}_X^\alpha\hat{B}_{\alpha\bar{\alpha}}-\hat{B}_{\alpha\bar{\alpha}}\hat{\mathcal{A}}_X^{\bar{\alpha}},$$

where $\hat{B}_{\alpha\bar{\alpha}}$ stands for an arbitrary $2\times(2N_b-2)$ matrix. As long as this matrix is a covariant quantity, i.e., $\hat{B}_{\alpha\bar{\alpha}}\to\hat{v}_\alpha^\dagger\hat{B}_{\alpha\bar{\alpha}}\hat{V}_{\bar{\alpha}}$, the interband derivative above clearly transforms in a covariant way under Eq. (37),

$$[\hat{D}_X^{\alpha\bar{\alpha}},\hat{B}_{\alpha\bar{\alpha}}]\to\hat{v}_\alpha^\dagger[\hat{D}_X^{\alpha\bar{\alpha}},\hat{B}_{\alpha\bar{\alpha}}]\hat{V}_{\bar{\alpha}}. \tag{41}$$

In terms of these quantities and derivatives, let us summarize henceforth several formulas which become useful in Sec. III D 2. Consider first a interband covariant derivative of a interband gauge field,

$$\begin{aligned}[\hat{D}_X^{\alpha\bar{\alpha}},\hat{\mathcal{A}}_{X'}^{\alpha\bar{\alpha}}]_{(\sigma|\beta\sigma')}&=[(\partial_X\hat{U}^\dagger)\partial_{X'}\hat{U}]_{(\alpha\sigma|\beta\sigma')}+[\hat{U}^\dagger\partial_{XX'}^2\hat{U}]_{(\alpha\sigma|\beta\sigma')}\\&-\sum_{\sigma''=\pm}[(\partial_X\hat{U}^\dagger)\hat{U}]_{(\alpha\sigma|\alpha\sigma'')}[\hat{U}^\dagger\partial_{X'}\hat{U}]_{(\alpha\sigma''|\beta\sigma')}\\&+\sum_{\gamma\neq\alpha}\sum_{\sigma''=\pm}[(\partial_{X'}\hat{U}^\dagger)\hat{U}]_{(\alpha\sigma|\gamma\sigma'')}[\hat{U}^\dagger\partial_X\hat{U}]_{(\gamma\sigma''|\beta\sigma')},\end{aligned}$$

where $\beta\neq\alpha$ and $\sigma,\sigma'=\pm$. Then, applying into the last two terms the following identity:

$$\sum_{\sigma=\pm}|\alpha\sigma\rangle\langle\alpha\sigma|=\hat{1}-\sum_{\gamma\neq\alpha}\sum_{\sigma=\pm}|\gamma\sigma\rangle\langle\gamma\sigma|,$$

we can exchange the subscripts of the covariant derivative and the gauge field with each other,

$$[\hat{D}_X^{\alpha\bar{\alpha}},\hat{\mathcal{A}}_{X'}^{\alpha\bar{\alpha}}]_{(\sigma|\beta\sigma')}\equiv[\hat{D}_{X'}^{\alpha\bar{\alpha}},\hat{\mathcal{A}}_X^{\alpha\bar{\alpha}}]_{(\sigma|\beta\sigma')}. \tag{42}$$

In the actual calculations, this equality becomes very powerful when combined with the "decomposition rule" of covariant derivatives such as

$$\begin{aligned}[\hat{D}_X^\alpha,\hat{B}_{\alpha\bar{\alpha}}\hat{B}'_{\bar{\alpha}}\hat{B}''_{\bar{\alpha}\alpha}]&=[\hat{D}_X^{\alpha\bar{\alpha}},\hat{B}_{\alpha\bar{\alpha}}]\hat{B}'_{\bar{\alpha}}\hat{B}''_{\bar{\alpha}\alpha}+\hat{B}_{\alpha\bar{\alpha}}[\hat{D}_X^{\bar{\alpha}},\hat{B}'_{\bar{\alpha}}]\hat{B}''_{\bar{\alpha}\alpha}\\&+\hat{B}_{\alpha\bar{\alpha}}\hat{B}'_{\bar{\alpha}}[\hat{D}_X^{\bar{\alpha}\alpha},\hat{B}''_{\bar{\alpha}\alpha}].\end{aligned} \tag{43}$$

One should also note that curvatures either in the αth band space or in its complementary space can be expressed also in terms of interband gauge fields,

$$\hat{\Omega}_{XX'}^\alpha=-i(\hat{\mathcal{A}}_X^{\alpha\bar{\alpha}}\hat{\mathcal{A}}_{X'}^{\bar{\alpha}\alpha}-\hat{\mathcal{A}}_{X'}^{\alpha\bar{\alpha}}\hat{\mathcal{A}}_X^{\bar{\alpha}\alpha}), \tag{44}$$

$$\hat{\Omega}_{XX'}^{\bar{\alpha}}=-i(\hat{\mathcal{A}}_X^{\bar{\alpha}\alpha}\hat{\mathcal{A}}_{X'}^{\alpha\bar{\alpha}}-\hat{\mathcal{A}}_{X'}^{\bar{\alpha}\alpha}\hat{\mathcal{A}}_X^{\alpha\bar{\alpha}}). \tag{45}$$

2. SU(2)×SU($2N_b-2$) *covariant manipulations*

Using the arithmetic described so far, we will study Eqs. (35) and (36) within the second order accuracy in λ. As is the case for a standard (such as Rayleigh-Schrödinger) perturbation theory, our second order expression for Eqs. (35) and (36) is given both by eigenenergies at the zeroth order and its eigen-wave-functions [see Eq. (55), for example]. Namely, differences between eigenvalues of $\hat{\mathsf{L}}$ enter into a sort of "energy denominator," while a "numerator" in a usual perturbation theory is now transcribed into a gauge field (connection), which is nothing but the matrix element of our perturbation part (i.e., $\partial_X\partial_Q$) among different eigenbases of

$\hat{L}$. Accordingly, just as in a usual perturbation theory, our second order expression for Eqs. (35) and (36) also depends on wave functions and eigenvalues not only for the αth band in question but also for the bands other than the αth band.

On the other hand, extensive semiclassical analyses in a noninteracting system[5–10,27] suggest that the low-energy effective theory for (quasi)particles should be constituted only by Bloch wave functions and energy dispersions for the αth band in question, while free from details of the other bands.

We shall show in this section that this is indeed the case for the SU(2) RKE given in Eqs. (35) and (36) at least up to second order in λ. To be more specific, we will transform Eq. (35) in combination with Eq. (36) into a more compact form rigorously up to $\mathcal{O}(\lambda^2)$, only to find that they are actually *given solely* in terms of the SU(2) gauge covariant quantities such as $\hat{\Omega}^{\alpha}_{XX'}$ and $[\hat{D}^{\alpha}_{X},\cdots]$. During this transformation, several formulas described in the previous section, such as Eqs. (42)–(45), become very useful.

To see this, let us begin with Eq. (35), i.e., the (α,α)th component (diagonal component) of the original dissipationless Keldysh equation,

$$-[\hat{L}_d,\hat{A}_\alpha]=\hat{F}^{(1)}_\alpha(\hat{A}_\alpha)+F^{(2)}_\alpha(\{\hat{A}_{\alpha\eta}\})+F^{(3)}_\alpha(\{\hat{A}_{\eta\delta}\}). \quad (46)$$

Here $\eta,\delta\neq\alpha$. For later clarity, the right hand side was decoupled with respect to different elements of $\hat{A}$. This becomes possible clearly because our differential equation is at most *linear* in $\hat{A}$. Up to first order in λ_1, the first term on the right hand side reads[28]

$$\hat{F}^{(1)}_\alpha(\hat{A}_\alpha)=\frac{i}{2}[[\hat{D}^{\alpha}_{X_j},\hat{L}_{d,\alpha}],[\hat{D}^{\alpha}_{Q_j},\hat{A}_\alpha]]_+-\{X_j\leftrightarrow Q_j\}$$
$$-[\hat{\mathcal{M}}_\alpha,\hat{A}_\alpha]_--\frac{1}{4}[\hat{\mathcal{N}}_\alpha,\hat{A}_\alpha]_++\mathcal{O}(\lambda_1^2), \quad (47)$$

where we have introduced, following 2×2 Hermite and anti-Hermite matrices, respectively,

$$\hat{\mathcal{M}}_\alpha\equiv\frac{i}{2}\{\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}(\hat{L}_{d,\bar{\alpha}}-L^{(0)}_{d,\alpha}\hat{1})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}-\{X_j\leftrightarrow Q_j\}\}. \quad (48)$$

$$\hat{\mathcal{N}}_\alpha\equiv[\hat{\epsilon}_\alpha,\hat{\Omega}^{\alpha}_{X_jQ_j}]_-. \quad (49)$$

$\hat{L}_{d,\bar{\alpha}}$ denotes a $(2N_b-2)\times(2N_b-2)$ diagonal block of $\hat{L}_d$,

$$\hat{L}_d\equiv\begin{bmatrix}\hat{L}_{d,\alpha} & \mathbf{0}\\ \mathbf{0} & \hat{L}_{d,\bar{\alpha}}\end{bmatrix}.$$

Observing Eq. (47), notice first the commutator between $\hat{\mathcal{M}}_\alpha$ and the αth band spectral function, i.e., $\hat{A}_\alpha$. This commutator implies that the former Hermitian matrix is the first order correction to the αth band dispersion. Notice also that $\hat{\mathcal{N}}_\alpha$ enters into the *anticommutator* with $\hat{A}_\alpha$. As will be shown later, this anticommutator lets $\hat{\mathcal{N}}_\alpha$ play a relevant role in determining the first order gradient expansion correction to the spectral weight (see Sec. IV for details).

The second order contributions in $\hat{F}^{(1)}_\alpha(\hat{A}_\alpha)$, $\hat{F}^{(2)}_\alpha(\hat{A}_{\alpha\bar{\alpha}})$, and $\hat{F}^{(3)}_\alpha(\hat{A}_{\bar{\alpha}})$ are given as follows:

$$\hat{F}^{(1)}(\hat{A}_\alpha)=\cdots-\frac{1}{8}\{\partial_{X_j}\partial_{X_k}L^{(0)}_{d,\alpha}\hat{1}+\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k}$$
$$+\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}\}\cdot\{[D^{\alpha}_{Q_j},[D^{\alpha}_{Q_k},\hat{A}_\alpha]]$$
$$+\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}\hat{A}_\alpha+\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}\}-\frac{1}{8}\{-\mathcal{A}^{\alpha\bar{\alpha}}_{X_k}[D^{\bar{\alpha}}_{X_j},L^{(0)}_{d,\alpha}\hat{1}$$
$$-\hat{L}_{d,\bar{\alpha}}]-\mathcal{A}^{\alpha\bar{\alpha}}_{X_j}[D^{\bar{\alpha}}_{X_k},L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}}]$$
$$-[D^{\alpha\bar{\alpha}}_{X_j},\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}](L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\}\cdot\{[D^{\bar{\alpha}\alpha}_{Q_j},\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}]\hat{A}_\alpha$$
$$+\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}[D^{\alpha}_{Q_j},\hat{A}_\alpha]+\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}[D^{\alpha}_{Q_k},\hat{A}_\alpha]\}$$
$$-\{X_j\leftrightarrow Q_j|X_k,Q_k\}-\{X_j,Q_j|X_k\leftrightarrow Q_k\}$$
$$+\{X_j\leftrightarrow Q_j|X_k\leftrightarrow Q_k\}-\text{H.c.}+\mathcal{O}(\lambda^3). \quad (50)$$

$$\hat{F}^{(2)}_\alpha(\hat{A}_{\alpha\bar{\alpha}})=\frac{i}{2}\{\partial_{X_j}L^{(0)}_{d,\alpha}(\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}\hat{A}_{\bar{\alpha}\alpha}-\hat{A}_{\alpha\bar{\alpha}}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j})-\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})$$
$$\times[D^{\bar{\alpha}\alpha}_{Q_j},\hat{A}_{\bar{\alpha}\alpha}]\}-\{X_j\leftrightarrow Q_j\}-\text{H.c.}+\mathcal{O}(\lambda^2), \quad (51)$$

$$\hat{F}^{(3)}_\alpha(\hat{A}_{\bar{\alpha}})=\frac{i}{2}\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}\hat{A}_{\bar{\alpha}}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}$$
$$-\{X_j\leftrightarrow Q_j\}-\text{H.c.}+\mathcal{O}(\lambda^2). \quad (52)$$

where "$\cdots$" in $\hat{F}^{(1)}$ stands for those terms explicit on the right hand side of Eq. (47). To rewrite the latter two, i.e., $\hat{F}^{(2)}_\alpha$ and $\hat{F}^{(3)}_\alpha$, only in terms of $\hat{A}_\alpha$, we first solve $\hat{A}_{\alpha\bar{\alpha}}$ and $\hat{A}_{\bar{\alpha}}$ in favor of $\hat{A}_\alpha$,

$$\hat{A}_{\alpha\bar{\alpha}}(\hat{A}_\alpha)=\frac{i}{2}\{-\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}(\partial_{X_k}L^{(0)}_{d,\alpha}\hat{1}+[D^{\bar{\alpha}}_{X_k},\hat{L}_{d,\bar{\alpha}}])$$
$$+[D^{\alpha}_{Q_k},\hat{A}_\alpha]\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}(\hat{L}_{d,\bar{\alpha}}-L^{(0)}_{d,\alpha}\hat{1})\}\cdot(\hat{L}_{d,\bar{\alpha}}-L^{(0)}_{d,\alpha}\hat{1})^{-1}$$
$$-\{X_k\leftrightarrow Q_k\}+\mathcal{O}(\lambda^2), \quad (53)$$

$$\hat{A}_{\eta\delta}(\hat{A}_\alpha)=\frac{i}{2}(\hat{L}_{d,\eta}\hat{\mathcal{A}}^{\eta\alpha}_{X_k}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\delta}_{Q_k}-\hat{\mathcal{A}}^{\eta\alpha}_{X_k}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\delta}_{Q_k}\hat{L}_{d,\delta})$$
$$\cdot(\hat{L}_{d,\delta}-\hat{L}_{d,\eta})^{-1}-\{X_k\leftrightarrow Q_k\}+\mathcal{O}(\lambda^2)$$
$$=-\frac{i}{2}\hat{\mathcal{A}}^{\eta\alpha}_{X_k}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\delta}_{Q_k}-\{X_k\leftrightarrow Q_k\}+\mathcal{O}(\lambda^2), \quad (54)$$

with $\eta,\delta\neq\alpha$. One can obtain these relations by looking into the interband components of the dissipationless Keldysh equation, precisely as in Eq. (36). We then substitute these two back into Eqs. (51) and (52), only to obtain $\hat{F}^{(2)}_\alpha$ and $\hat{F}^{(3)}_\alpha$ as a functional of $\hat{A}_\alpha$,

$$\hat{F}_{\alpha}^{(2)}(\hat{A}_{\alpha}) = \frac{1}{2}(\partial_{X_j} L_{d,\alpha}^{(0)})\hat{A}_{\alpha}\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} + \hat{L}_{d,\bar{\alpha}}][L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}]^{-1}\hat{\mathcal{A}}_{Q_j}^{\bar{\alpha}\alpha} + \frac{1}{2}(\partial_{X_j} L_{d,\alpha}^{(0)})[\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}\hat{\mathcal{A}}_{Q_j}^{\bar{\alpha}\alpha}$$

$$- \frac{1}{4}[\hat{D}_{Q_j}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} + \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha} - \frac{1}{4}\hat{A}_{\alpha}[\hat{D}_{Q_j}^{\alpha\bar{\alpha}}, \hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}][\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} + \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha}$$

$$- \frac{1}{4}\hat{A}_{\alpha}\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}[\hat{D}_{Q_j}^{\bar{\alpha}}, [\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} + \hat{L}_{d,\bar{\alpha}}]]\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha} - \frac{1}{4}[\hat{D}_{Q_j}^{\alpha}, [\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}]]\hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}(L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha}$$

$$- \frac{1}{4}[\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}][\hat{D}_{Q_j}^{\alpha\bar{\alpha}}, \hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha} - \frac{1}{4}[\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}[\hat{D}_{Q_j}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha}$$

$$- \frac{1}{4}\hat{A}_{\alpha}\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} + \hat{L}_{d,\bar{\alpha}}][\hat{D}_{Q_j}^{\bar{\alpha}}, (L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})^{-1}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha}$$

$$+ \frac{1}{4}[\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}(\hat{L}_{d,\bar{\alpha}} - L_{d,\alpha}^{(0)}\hat{1})[\hat{D}_{Q_j}^{\bar{\alpha}}, (L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})^{-1}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha} - \{X_j \leftrightarrow Q_j | X_k, Q_k\}$$

$$- \{X_j, Q_j | X_k \leftrightarrow Q_k\} + \{X_j \leftrightarrow Q_j | X_k \leftrightarrow Q_k\} - \text{H.c.} + \mathcal{O}(\lambda^3), \tag{55}$$

and

$$\hat{F}_{\alpha}^{(3)}(\hat{A}_{\alpha}) = \frac{1}{4}\hat{\mathcal{A}}_{Q_j}^{\alpha\bar{\alpha}}\hat{\mathcal{A}}_{X_k}^{\bar{\alpha}\alpha}\hat{A}_{\alpha}\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}(L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha}$$

$$- \{X_j \leftrightarrow Q_j | X_k, Q_k\} - \{X_j, Q_j | X_k \leftrightarrow Q_k\}$$

$$+ \{X_j \leftrightarrow Q_j | X_k \leftrightarrow Q_k\} - \text{H.c.} + \mathcal{O}(\lambda^3). \tag{56}$$

Equations (50), (55), and (56), in combination with Eq. (47), are all the terms that enter into the right hand side of Eq. (46) up to $\mathcal{O}(\lambda_1^2, \lambda_1\lambda_2, \lambda_2^2)$.

Now that we have obtained an explicit form of Eq. (46), we shall next find out a 2×2 matrix $\hat{A}_{\alpha}$ that satisfies this equation. Before doing this, however, it would be clearly helpful to simplify these more than 100 terms. In fact, we can further transform all these terms precisely into Eqs. (67)–(69), whose transparent expression helps us to find $\hat{A}_{\alpha}$ in Sec. IV. To be more specific, we can find, for any single term in Eqs. (50), (55), and (56), several counterpart terms with which it constitutes a certain SU(2) gauge covariant quantity enumerated in Eqs. (67)–(69).

Terms proportional to $[\hat{D}_Q^{\alpha}, \hat{A}_{\alpha}]$. To see this explicitly, focus first on those terms in Eqs. (50), (55), and (56) that are linear in $[\hat{D}^{\alpha}, \hat{A}_{\alpha}]$;

$$\frac{1}{8}\{\mathcal{A}_{X_k}^{\alpha\bar{\alpha}}[D_{X_j}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}] + \mathcal{A}_{X_j}^{\alpha\bar{\alpha}}[D_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}]$$

$$+ [D_{X_j}^{\alpha\bar{\alpha}}, \hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\}$$

$$\times\{\hat{\mathcal{A}}_{Q_k}^{\bar{\alpha}\alpha}[D_{Q_j}^{\alpha}, \hat{A}_{\alpha}] + \hat{\mathcal{A}}_{Q_j}^{\bar{\alpha}\alpha}[D_{Q_k}^{\alpha}, \hat{A}_{\alpha}]\}$$

$$+ \frac{1}{2}(\partial_{X_j} L_{d,\alpha}^{(0)})[\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}\hat{\mathcal{A}}_{Q_j}^{\bar{\alpha}\alpha}$$

$$- \frac{1}{4}[\hat{D}_{Q_j}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} + \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha}$$

$$- \frac{1}{4}[\hat{D}_{Q_k}^{\alpha}, \hat{A}_{\alpha}][\hat{D}_{Q_j}^{\alpha\bar{\alpha}}, \hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{X_j}^{\bar{\alpha}\alpha} + \cdots, \tag{57}$$

where we have already used $[\hat{D}_Q^{\bar{\alpha}}, (L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})^{-1}]\cdot(L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}) = -(L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})^{-1}\cdot[\hat{D}_Q^{\bar{\alpha}}, (L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})]$. Note also that "$\cdots$" above indicates those three kinds of *counterpart terms* with X and Q exchanged and their *Hermite conjugate terms*,

$$\text{"}\cdots\text{"} \equiv -\{X_j \leftrightarrow Q_j | X_k, Q_k\} - \{X_j, Q_j | X_k \leftrightarrow Q_k\}$$

$$+ \{X_j \leftrightarrow Q_j | X_k \leftrightarrow Q_k\} - \text{H.c.} \tag{58}$$

When taking these terms implicit into account, we may rewrite the first term in Eq. (57) into the following:

$$\frac{1}{8}\{\mathcal{A}_{X_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_j}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}] + \mathcal{A}_{X_j}^{\alpha\bar{\alpha}}[\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}]$$

$$+ [\hat{D}_{X_j}^{\alpha\bar{\alpha}}, \hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\}$$

$$\times\{\hat{\mathcal{A}}_{Q_k}^{\bar{\alpha}\alpha}[D_{Q_j}^{\alpha}, \hat{A}_{\alpha}] + \hat{\mathcal{A}}_{Q_j}^{\bar{\alpha}\alpha}[D_{Q_k}^{\alpha}, \hat{A}_{\alpha}]\} + \cdots$$

$$= -\frac{1}{4}[\hat{D}_{Q_j}^{\alpha}, \hat{A}_{\alpha}]\hat{\mathcal{A}}_{Q_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_k}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}]\mathcal{A}_{X_j}^{\bar{\alpha}\alpha}$$

$$+ \frac{1}{4}\mathcal{A}_{X_k}^{\alpha\bar{\alpha}}[\hat{D}_{X_j}^{\bar{\alpha}}, L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}_{Q_k}^{\bar{\alpha}\alpha}[\hat{D}_{Q_j}^{\alpha}, \hat{A}_{\alpha}]$$

$$+ \frac{1}{4}[\hat{D}_{X_j}^{\alpha\bar{\alpha}}, \hat{\mathcal{A}}_{X_k}^{\alpha\bar{\alpha}}](L_{d,\alpha}^{(0)}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}_{Q_k}^{\bar{\alpha}\alpha}[\hat{D}_{Q_j}^{\alpha}, \hat{A}_{\alpha}] + \cdots.$$

Namely, we replaced several terms in the left hand side by either their counterparts or their Hermitian conjugates implicit in "$\cdots$." We also used Eq. (42), only to obtain the third term in the right hand side. Since we can regard that these terms were just swapped among their three other copies and their Hermitian conjugates, we can begin with the following, instead of Eq. (57):

$$\text{Eq. (57)} = \frac{1}{2}(\partial_{X_j} L^{(0)}_{d,\alpha})[D^{\alpha}_{Q_k}, \hat{A}_\alpha](\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j} - \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k})$$
$$+ \frac{1}{4}\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}(L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}})[D^{\bar{\alpha}\alpha}_{Q_j}, \hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k}][D^{\alpha}_{Q_k}, \hat{A}_\alpha]$$
$$+ \frac{1}{4}\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}[D^{\bar{\alpha}}_{X_j}, L_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}[D^{\alpha}_{Q_j}, \hat{A}_\alpha]$$
$$+ \frac{1}{4}[D^{\alpha\bar{\alpha}}_{X_j}, \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}](L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}[D^{\alpha}_{Q_j}, \hat{A}_\alpha] + \cdots,$$

where "⋯" is defined in Eq. (58).

Compare the second, third, and fourth terms above with the decomposition rule for derivatives, i.e., Eq. (43). Namely, we can unify these three into a single αth band covariant derivative term,

$$\text{Eq. (57)} = -\frac{i}{2}(\partial_{X_j} L^{(0)}_{d,\alpha})[D^{\alpha}_{Q_k}, \hat{A}_\alpha]\hat{\Omega}^{\alpha}_{Q_j X_k}$$
$$+ \frac{1}{4}[\hat{D}^{\alpha}_{X_j}, \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}(L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}][D^{\alpha}_{Q_j}, \hat{A}_\alpha] + \cdots. \tag{59}$$

When combined explicitly with its counterpart with X_k and Q_k exchanged, the second term above is expressed by $\hat{\mathcal{M}}_\alpha$ [see Eq. (48)]. Accordingly, Eq. (57) can be rigorously transformed into the following SU(2) covariant quantities:

$$\text{Eq. (57)} = -\frac{i}{2}(\partial_{X_j} L^{(0)}_{d,\alpha})[\hat{\Omega}^{\alpha}_{Q_j X_k}, [\hat{D}^{\alpha}_{Q_k}, \hat{A}_\alpha]]_+ - \{X_j \leftrightarrow Q_j | X_k, Q_k\}$$
$$- \{X_j, Q_j | X_k \leftrightarrow Q_k\} + \{X_j \leftrightarrow Q_j | X_k \leftrightarrow Q_k\}$$
$$- \frac{i}{2}[[\hat{D}^{\alpha}_{X_j}, \hat{\mathcal{M}}_\alpha], [\hat{D}^{\alpha}_{Q_j}, \hat{A}_\alpha]]_+ - \{Q_j \leftrightarrow X_j\}. \tag{60}$$

Terms proportional to $\hat{A}_\alpha$. Employing a similar manipulation, we can further simplify all the remaining terms which are linear in either (a) $\hat{A}_\alpha$ itself or (b) $[D^{\alpha}_{Q_j}, [D^{\alpha}_{Q_k}, \hat{A}_\alpha]]$. As will be shown next, the latter one can be easily proven to be zero in total up to the order of $\mathcal{O}(\lambda^2)$. Thus, we will henceforth look into those terms in Eqs. (50), (55), and (56) which are proportional to $\hat{A}_\alpha$,

$$-\frac{1}{8}\{\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}(L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k} + (X_j \leftrightarrow X_k)\}$$
$$\times\{\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}\hat{A}_\alpha + \hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}\}$$
$$+ \frac{1}{8}\{\mathcal{A}^{\alpha\bar{\alpha}}_{X_k}[D^{\bar{\alpha}}_{X_j}, L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}}] + (X_j \leftrightarrow X_k)\}[D^{\bar{\alpha}\alpha}_{Q_j}, \hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}]\hat{A}_\alpha$$
$$- \frac{1}{4}\hat{A}_\alpha[\hat{D}^{\alpha\bar{\alpha}}_{Q_j}, \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}][\hat{D}^{\bar{\alpha}}_{X_k}, L^{(0)}_{d,\alpha}\hat{1} + \hat{L}_{d,\bar{\alpha}}]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}$$
$$- \frac{1}{4}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}[\hat{D}^{\bar{\alpha}}_{Q_j}, [\hat{D}^{\bar{\alpha}}_{X_k}, L^{(0)}_{d,\alpha}\hat{1} + \hat{L}_{d,\bar{\alpha}}]]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}$$
$$+ \frac{1}{4}\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}(L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j} + \cdots, \tag{61}$$

where "⋯" is defined in Eq. (58). We have already neglected those which are canceled either by their counterparts or by their hermite conjugates.

Notice first that, to second order in λ, we can regard that $\hat{A}_\alpha$ in Eq. (61) commutes with other 2×2 matrices. This is because $\hat{A}_\alpha$ reduces to a unit matrix at equilibrium, while Eq. (61) is already at least second order in λ_1. Thus, commutators between $\hat{A}_\alpha$ and other 2×2 matrices inevitably end up with the third order contributions in λ, e.g.,

$$(\partial_{X_k} L^{(0)}_{d,\alpha})\hat{A}_\alpha[\hat{D}^{\alpha\bar{\alpha}}_{Q_k}, \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}$$
$$= (\partial_{X_k} L^{(0)}_{d,\alpha})[\hat{D}^{\alpha\bar{\alpha}}_{Q_k}, \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}\hat{A}_\alpha + \mathcal{O}(\lambda_1^2\lambda_2, \lambda_1^3). \tag{62}$$

Observing this, one can easily unify the second term and (the Hermitian conjugate of) third term in Eq. (61), so that they are given solely in terms of the Berry's curvature for the αth band,

$$\frac{1}{8}\{\mathcal{A}^{\alpha\bar{\alpha}}_{X_k}[D^{\bar{\alpha}}_{X_j}, L^{(0)}_{d,\alpha}\hat{1} - \hat{L}_{d,\bar{\alpha}}] + (X_j \leftrightarrow X_k)\}[D^{\bar{\alpha}\alpha}_{Q_j}, \hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}]\hat{A}_\alpha$$
$$+ \frac{1}{4}\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}[\hat{D}^{\bar{\alpha}}_{X_k}, L^{(0)}_{d,\alpha}\hat{1} + \hat{L}_{d,\bar{\alpha}}][\hat{D}^{\bar{\alpha}\alpha}_{Q_j}, \hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_k}]\hat{A}_\alpha$$
$$= \frac{1}{2}(\partial_{X_k} L^{(0)}_{d,\alpha})\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}[\hat{D}^{\bar{\alpha}\alpha}_{Q_k}, \hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}]\hat{A}_\alpha.$$

Namely, when combined with its counterpart with X_j and Q_j exchanged and their Hermitian conjugate, $\mathcal{O}(\lambda^2)$ contribution of the right hand side above can be expressed only by $\hat{\Omega}^{\alpha}_{X_j Q_j}$,

$$\frac{1}{2}(\partial_{X_k} L^{(0)}_{d,\alpha})\{\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}[\hat{D}^{\bar{\alpha}\alpha}_{Q_k}, \hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}]\hat{A}_\alpha + \hat{A}_\alpha[\hat{D}^{\alpha\bar{\alpha}}_{Q_k}, \hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j}\}$$
$$- \{X_j \leftrightarrow Q_j | X_k, Q_k\} = \frac{i}{2}(\partial_{X_k} L^{(0)}_{d,\alpha})[\hat{D}^{\alpha}_{Q_k}, \hat{\Omega}^{\alpha}_{X_j Q_j}]\hat{A}_\alpha + \mathcal{O}(\lambda^3), \tag{63}$$

where we used Eqs. (62), (43), and (44).

The first, fourth, and fifth terms in Eq. (61) are described in terms of $\hat{\mathcal{M}}_\alpha$ and $\hat{\Omega}_{X_j Q_j}$ in total. To see this, let us begin with the fourth term,

$$
\begin{aligned}
&-\frac{1}{4}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}[\hat{D}^{\bar{\alpha}}_{Q_j},[\hat{D}^{\bar{\alpha}}_{X_k},L^{(0)}_{d,\alpha}\hat{1}+\hat{L}_{d,\bar{\alpha}}]]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}+\cdots\\
&\quad=-\frac{1}{8}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}[\hat{D}^{\bar{\alpha}}_{Q_j},[\hat{D}^{\bar{\alpha}}_{X_k},L^{(0)}_{d,\alpha}\hat{1}+\hat{L}_{d,\bar{\alpha}}]]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}\\
&\qquad+\frac{1}{8}\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}[\hat{D}^{\bar{\alpha}}_{X_j},[\hat{D}^{\bar{\alpha}}_{Q_k},L^{(0)}_{d,\alpha}\hat{1}+\hat{L}_{d,\bar{\alpha}}]]\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k}\hat{A}_\alpha+\cdots\\
&\quad=-\frac{1}{8}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}([\hat{D}^{\bar{\alpha}}_{Q_j},[\hat{D}^{\bar{\alpha}}_{X_k},L^{(0)}_{d,\alpha}\hat{1}+\hat{L}_{d,\bar{\alpha}}]]\\
&\qquad-[\hat{D}^{\bar{\alpha}}_{X_k},[\hat{D}^{\bar{\alpha}}_{Q_j},L^{(0)}_{d,\alpha}\hat{1}+\hat{L}_{d,\bar{\alpha}}]])\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}+\mathcal{O}(\lambda^3)+\cdots\\
&\quad=\frac{i}{8}\hat{A}_\alpha\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}[\hat{\Omega}^{\bar{\alpha}}_{X_kQ_j},L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}}]_-\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}+\cdots\\
&\quad=\frac{1}{8}\hat{A}_\alpha[\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k},\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}]_-\\
&\qquad-\frac{1}{8}\hat{A}_\alpha[\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{Q_j},\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}]_-+\cdots,
\end{aligned}
\tag{64}
$$

where "⋯" stands for three other counterparts and their Hermitian conjugates. In the first equality, we have swapped 1/2 of the first term on the left hand side for its counterpart in (the Hermitian conjugates of) $\{X_j\leftrightarrow Q_j|X_k\leftrightarrow Q_k\}$. In the second equality, we have ignored $\mathcal{O}(\lambda_1^2\lambda_2,\lambda_1^3)$ contribution associated with the commutators between $\hat{A}_\alpha$ and other 2×2 matrices. We further used Eqs. (39) and (45), only to reach the final expression.

Within $\mathcal{O}(\lambda^2)$, the second term in Eq. (64) is canceled by the first and fifth terms in Eq. (61),

$$
\begin{aligned}
&(1\text{st})+(4\text{th})+(5\text{th})\text{ in Eq. (61)}\\
&\quad=\frac{1}{8}\hat{A}_\alpha[\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_k}\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k},\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{Q_j}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}]_-+\mathcal{O}(\lambda^3)+\cdots.
\end{aligned}
\tag{65}
$$

Then notice that, when combined with its three other counterparts implicit in "⋯," the coefficient of $\hat{A}_\alpha$ above clearly reduces to the commutator between $\hat{\Omega}^\alpha_{Q_kX_k}$ and $\hat{\mathcal{M}}_\alpha$ [see Eqs. (44) and (48)]. In combination with Eq. (63), this dictates that $\mathcal{O}(\lambda^2)$ contributions in Eq. (61) are indeed given only by $\hat{\mathcal{M}}_\alpha$ and $\hat{\Omega}^\alpha_{X_jQ_j}$,

$$
\begin{aligned}
\text{Eq. (61)}=&\frac{i}{4}(\partial_{X_k}L^{(0)}_{d,\alpha})[[\hat{D}^\alpha_{Q_k},\hat{\Omega}^\alpha_{X_jQ_j}],\hat{A}_\alpha]_+-\{X_k\leftrightarrow Q_k\}\\
&-\frac{1}{4}[[\hat{\mathcal{M}}_\alpha,\hat{\Omega}^\alpha_{X_jQ_j}]_-,\hat{A}_\alpha]_+\mathcal{O}(\lambda^3).
\end{aligned}
\tag{66}
$$

Terms proportional to $[\hat{D}^\alpha_Q,[\hat{D}^\alpha_{Q'},\hat{A}_\alpha]]$. Those terms in Eqs. (50), (55), and (56) which are linear in the second covariant derivative vanish up to $\mathcal{O}(\lambda^2)$. One can easily see this by noting that any commutator between $\hat{A}_\alpha$ and other matrices in Eqs. (50), (55), and (56) ends up with $\mathcal{O}(\lambda_1^2\lambda_2,\lambda_1^3)$,

$$
\begin{aligned}
&-\frac{1}{8}\{\partial_{X_j}\partial_{X_k}L^{(0)}_{d,\alpha}\hat{1}+\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k}\\
&\quad+\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}\}\cdot[D^\alpha_{Q_j},[D^\alpha_{Q_k},\hat{A}_\alpha]]\\
&\quad-\frac{1}{4}[\hat{D}^\alpha_{Q_j},[\hat{D}^\alpha_{Q_k},\hat{A}_\alpha]]\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_k}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_j}+\cdots\\
&\quad=-\frac{1}{8}\{\hat{\mathcal{A}}^{\alpha\bar{\alpha}}_{X_j}(L^{(0)}_{d,\alpha}\hat{1}-\hat{L}_{d,\bar{\alpha}})\hat{\mathcal{A}}^{\bar{\alpha}\alpha}_{X_k}-\{X_k\leftrightarrow X_j\}\}\partial_{Q_j}\partial_{Q_k}(\hat{A}_\alpha)\\
&\qquad+\mathcal{O}(\lambda^3)+\cdots=\mathcal{O}(\lambda^3),
\end{aligned}
$$

where "⋯" is already defined in Eq. (58).

3. SU(2) *reduced Keldysh equation up to the second order accuracy and its physical implications*

According to the analyses up to the previous section such as Eqs. (47), (60), and (66), the $\mathcal{O}(\lambda^2)$ contributions of Eqs. (50), (55), and (56) can be rigorously transformed into the following:

$$
\begin{aligned}
\hat{\mathcal{L}}_1(\hat{A}_\alpha)+\hat{\mathcal{L}}_2(\hat{A}_\alpha)\equiv&[\hat{L}_{d,\alpha},\hat{A}_\alpha]+\hat{F}^{(1)}_\alpha(\hat{A}_\alpha)+\hat{F}^{(2)}_\alpha(\hat{A}_\alpha)\\
&+\hat{F}^{(3)}_\alpha(\hat{A}_\alpha)+\mathcal{O}(\lambda^3),
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\hat{\mathcal{L}}_1(\hat{A}_\alpha)=&-[\hat{\epsilon}_\alpha+\hat{\mathcal{M}}_\alpha,\hat{A}_\alpha]_-+i(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_k},\hat{A}_\alpha]\\
&-i(\partial_{Q_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{X_k},\hat{A}_\alpha],
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
\hat{\mathcal{L}}_2(\hat{A}_\alpha)=&-\frac{1}{4}[[\hat{\epsilon}_\alpha,\hat{\Omega}^\alpha_{X_j,Q_j}]_-,\hat{A}_\alpha]_+-\frac{1}{4}[[\hat{\mathcal{M}}_\alpha,\hat{\Omega}^\alpha_{X_j,Q_j}]_-,\hat{A}_\alpha]_+\\
&+\frac{1}{4}[i(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_k},\hat{\Omega}^\alpha_{X_j,Q_j}]-i(\partial_{Q_k}L^{(0)}_{d,\alpha})\\
&\times[\hat{D}^\alpha_{X_k},\hat{\Omega}^\alpha_{X_j,Q_j}],\hat{A}_\alpha]_+-\frac{i}{2}[[\hat{D}^\alpha_{X_j},\hat{\epsilon}_\alpha+\hat{\mathcal{M}}_\alpha],[\hat{D}^\alpha_{Q_j},\hat{A}_\alpha]]_+\\
&+\frac{i}{2}[[\hat{D}^\alpha_{Q_j},\hat{\epsilon}_\alpha+\hat{\mathcal{M}}_\alpha],[\hat{D}^\alpha_{X_j},\hat{A}_\alpha]]_+\\
&-\frac{i}{2}(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{\Omega}^\alpha_{Q_k,X_j},[\hat{D}^\alpha_{Q_j},\hat{A}_\alpha]]_+\\
&+\frac{i}{2}(\partial_{Q_k}L^{(0)}_{d,\alpha})[\hat{\Omega}^\alpha_{X_k,X_j},[\hat{D}^\alpha_{Q_j},\hat{A}_\alpha]]_+\\
&+\frac{i}{2}(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{\Omega}^\alpha_{Q_k,Q_j},[\hat{D}^\alpha_{X_j},\hat{A}_\alpha]]_+\\
&-\frac{i}{2}(\partial_{Q_k}L^{(0)}_{d,\alpha})[\hat{\Omega}^\alpha_{X_k,Q_j},[\hat{D}^\alpha_{X_j},\hat{A}_\alpha]]_+.
\end{aligned}
\tag{69}
$$

$\mathcal{L}_i$ above stands for the $\mathcal{O}(\lambda^i)$ contribution. Observing these results, notice that the first term in $\hat{\mathcal{L}}_2(\hat{\mathcal{A}}_\alpha)$ is nothing but the anticommutator between $\hat{\mathcal{N}}_\alpha$ and $\hat{A}_\alpha$, encoded in $\mathcal{O}(\lambda_1)$ contribution of $F^{(1)}(\hat{A}_\alpha)$ [see Eqs. (47) and (49)]. On the one hand, the second term in $\hat{\mathcal{L}}_2(\hat{\mathcal{A}}_\alpha)$ is obtained from the $\mathcal{O}(\lambda_1^2)$ contribution in $\hat{F}^{(1)}$, $\hat{F}^{(2)}$, and $\hat{F}^{(3)}$, i.e., Eq. (66). In spite of

these apparently different origins, $\hat{\epsilon}_\alpha$ and $\hat{\mathcal{M}}_\alpha$ in these first two terms appear in a totally *parallel* fashion. We can see a similar feature also in the fifth and sixth terms in $\mathcal{L}_2(\hat{A}_\alpha)$. These observations not only imply the consistency of our derived SU(2) RKE but also dictate that $\hat{\mathcal{M}}_\alpha$ indeed plays role of the $\mathcal{O}(\lambda_1)$ correction to the quasiparticle energy dispersion. Therefore, its nontrivial matrix structure, as well as that of $\hat{\epsilon}_\alpha$, induces the spin precession of quasiparticles.

When a disequilibration is created only by the external electromagnetic fields, $\hat{\mathcal{M}}_\alpha$ defined in Eq. (48) is composed by the *spatial* (magnetic) component and *temporal* (electric) one. The former is the Zeeman coupling between an external magnetic field and internal magnetic moment associated quasiparticle wave packet. The latter one is that between an applied electric field and internal electric dipole moment. To see these internal dipole moments explicitly, consider a situation in the presence of physical (i.e., external) electromagnetic gauge fields $(a_0, \mathbf{a})$, with corresponding physical electromagnetic fields $\mathbf{b}=\nabla\times\mathbf{a}$, $\mathbf{e}=\nabla_R a_0-\partial_T\mathbf{a}$. Then, following the standard recipe, let us introduce the canonical momentum $k\equiv q+\mathbf{a}(T,R)$ and canonical frequency $\omega'\equiv\omega-a_0(R)$,[29]

$$|u^{\alpha\sigma}_{X;Q}\rangle \equiv |u^{(0),\alpha\sigma}_{k\equiv q+a,\omega'\equiv\omega-a_0}\rangle.$$

Substituting this into Eq. (48), one can then reexpress $\hat{\mathcal{M}}_\alpha$ such that it is given solely by the partial derivative with respect to these canonical quantities. Namely, this $\mathcal{O}(\lambda_1)$ correction to a qp energy dispersion precisely reduces into an inner product between external electromagnetic fields and sort of internal dipole moments,

$$[\hat{\mathcal{M}}_\alpha] = \mathbf{b}\cdot[\hat{\mathbf{M}}_\alpha] + \mathbf{e}\cdot[\hat{\mathbf{P}}_\alpha]$$

$$[\hat{\mathbf{M}}_{\alpha,m}]_{\sigma\sigma'} = \frac{i\epsilon_{mnl}}{2}\langle\partial_{k_n}u^{(0),\alpha\sigma}_{k,\omega'}|L^{(0)}_{d,\alpha}-\hat{\mathsf{L}}|\partial_{k_l}u^{(0),\alpha\sigma'}_{k,\omega'}\rangle, \quad (70)$$

$$[\hat{\mathbf{P}}_{\alpha,m}]_{\sigma\sigma'} = \frac{i}{2}(\langle\partial_{\omega'}u^{(0),\alpha\sigma}_{k,\omega'}|L^{(0)}_{d,\alpha}-\hat{\mathsf{L}}|\partial_{k_m}u^{(0),\alpha\sigma'}_{k,\omega'}\rangle - \langle\partial_{k_m}u^{(0),\alpha\sigma}_{k,\omega'}|L^{(0)}_{d,\alpha}-\hat{\mathsf{L}}|\partial_{\omega'}u^{(0),\alpha\sigma'}_{k,\omega'}\rangle). \quad (71)$$

Note that the magnetic dipole moment $\mathbf{M}_\alpha$ above reproduces Eq. (4) in a noninteracting limit. On the other hand, the electric dipole moment $\mathbf{P}_\alpha$ has no noninteracting counterpart, since it is purely associated with the energy derivative of the quasiparticle Bloch wave function.

As is trivial from its coupling with $\mathbf{e}$, when considered in U(1) FLs, this electric dipole moment $\mathbf{P}_\alpha$ is literally time-reversally even while parity odd,

$$\mathbf{P}_\alpha(k) = \mathbf{P}_\alpha(-k), \quad T \text{ reversal},$$

$$\mathbf{P}_\alpha(k) = -\mathbf{P}_\alpha(-k), \quad I \text{ inverse}.$$

Therefore, in SU(2) FLs where both of these symmetries are guaranteed, $\mathbf{P}_\alpha$ becomes traceless just in a same way as $\mathbf{M}_\alpha$ does,

$$\mathrm{Tr}[\mathbf{P}_\alpha(k)] \equiv \mathrm{Tr}[\mathbf{M}_\alpha(k)] \equiv 0. \quad (72)$$

When translated into the SU(2) effective EOM, these two dipole moments, first of all, enter into that for the CP^1 vector,

$$i\frac{d\mathbf{z}}{dt} = \{\hat{\mathbf{M}}_\alpha\cdot\mathbf{b}+\hat{\mathbf{P}}_\alpha\cdot\mathbf{e}+\cdots\}\mathbf{z},$$

which describes the spin precession due to the external electromagnetic fields. Observing Eqs. (67)–(69), note also that every derivative term found in a conventional Keldysh equation is now replaced by the corresponding SU(2) covariant derivative in our SU(2) RKE,

$$\partial_X(\cdots) \to [\hat{D}^\alpha_X, \cdots] \equiv \partial_X(\cdots) + [\hat{\mathcal{A}}^\alpha_X, \cdots].$$

Therein, its usual derivative part describes the charge degree of freedom, while the commutator with $\hat{\mathcal{A}}^\alpha_X$ stands for the precession of quasiparticle spin due to this gauge field. In this sense, our derived RKE treats the charge and spin degrees of freedom on a *equal footing way* by using the SU(2) covariant derivative.

To uncover the physical significance of remaining terms in $\hat{\mathcal{L}}_2$ such as the first four terms and the final four terms, we further need to solve this reduced Keldysh equation in favor of $\hat{A}_\alpha$ (Sec. IV) and to derive the effective Boltzmann equation (Sec. V), respectively.

IV. PERTURBATIVE SOLUTION FOR REDUCED KELDYSH EQUATIONS

A. Prescription

When obtaining the reduced Keldysh equation described in the previous section, we have chosen the spectral function at equilibrium as a unit matrix with its coefficient to be a delta function of $L^{(0)}_{d,\alpha}$,

$$\hat{A}_\alpha = \delta(L^{(0)}_{d,\alpha})\hat{1} = Z^{(0)}_\alpha\delta(\omega-\epsilon^{(0)}_\alpha)\hat{1} \equiv f_0(L^{(0)}_{d,\alpha})\hat{1}. \quad (73)$$

This is not only because such solutions are physically sensible, but also because starting from this zeroth order spectral functions, we can indeed keep on satisfying the RKE up to higher order in λ. We will prove this point by solving actually the derived RKE in favor for $\hat{A}_\alpha$ perturbatively in λ [see Eq. (81)].

Our choice of the spectral function at equilibrium clearly satisfies the derived SU(2) RKE up to the first order,

$$\hat{\mathcal{L}}_1(f_0\hat{1}) = i\{(\partial_{X_k}L^{(0)}_{d,\alpha})(\partial_{Q_k}L^{(0)}_{d,\alpha}) - \{X_k\leftrightarrow Q_k\}\}f_0'\hat{1} = \hat{0}. \quad (74)$$

However, it does not up to the second order, i.e., $\hat{\mathcal{L}}_2(f_0\hat{1}) \neq \hat{0}$. To resolve this, we will introduce the first order correction to the spectral functions,

$$\hat{A}_\alpha = f_0\hat{1} + \hat{f}_1 + \cdots, \quad (75)$$

such that $\hat{A}_\alpha$ satisfies the SU(2) RKEs even up to the second order accuracy,

$$\hat{0} = \hat{\mathcal{L}}_1(\hat{f}_1) + \hat{\mathcal{L}}_2(f_0\hat{1}). \tag{76}$$

In general, we could further obtain the higher order correction of the spectral function $\hat{A}_\alpha$, provided that the RKE is given up to the third order's accuracy or higher than that, i.e., $\hat{\mathcal{L}}=\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2+\hat{\mathcal{L}}_3+\hat{\mathcal{L}}_4+\cdots$. Namely, its third order part $\hat{\mathcal{L}}_3$ determines the second order correction to $\hat{A}_\alpha$ in terms of $\hat{f}_1$ and f_0,

$$\hat{0} = \hat{\mathcal{L}}_1(\hat{f}_2) + \hat{\mathcal{L}}_2(\hat{f}_1) + \hat{\mathcal{L}}_3(f_0\hat{1}). \tag{77}$$

The following equations further specify the higher order corrections such as $\hat{f}_3,\ldots$, in terms of $\hat{f}_2$, $\hat{f}_1$, and f_0 iteratively;

$$\hat{0} = \hat{\mathcal{L}}_1(\hat{f}_3) + \hat{\mathcal{L}}_2(\hat{f}_2) + \hat{\mathcal{L}}_3(\hat{f}_1) + \hat{\mathcal{L}}_4(f_0\hat{1}),$$

$$\hat{0} = \cdots. \tag{78}$$

However, our derived RKE being exact up to $\mathcal{O}(\lambda_1^2,\lambda_1\lambda_2,\lambda_2^2)$, the highest order to which accuracy we could determine $\hat{A}_\alpha$, is $\hat{f}_1$.

B. First order correction to $\hat{A}_\alpha$

As we will show below, even up to this lowest order analysis, we obtain a nontrivial correction to the spectral function, having an interesting physical implication (next section). Notice first the last four terms in Eq. (69) do not contribute at all to $\mathcal{L}_2(f_0\hat{1})$, since they are antisymmetrized with respect to the exchange between X and Q,

$$\begin{aligned}\mathcal{L}_2(f_0\hat{1}) = &-\frac{f_0}{2}\{[\hat{\epsilon}_\alpha+\hat{\mathcal{M}}_\alpha,\hat{\Omega}^\alpha_{X_jQ_j}]_- - i(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_k},\hat{\Omega}^\alpha_{X_jQ_j}]\\ &+ i(\partial_{Q_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{X_k},\hat{\Omega}^\alpha_{X_jQ_j}]\} - f_0'\{i(\partial_{Q_j}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{X_j},\hat{\epsilon}_{t,\alpha}]\\ &- i(\partial_{X_j}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_j},\hat{\epsilon}_{t,\alpha}]\},\end{aligned}$$

where $\hat{\epsilon}_{t,\alpha}\equiv\hat{\epsilon}_\alpha+\hat{\mathcal{M}}_\alpha$. Then, the last two terms above are readily canceled in Eq. (76) when $\hat{f}_1=-\hat{\epsilon}_{t,\alpha}f_0'$ is substituted into $\hat{\mathcal{L}}_1$,

$$\begin{aligned}&\hat{\mathcal{L}}_1(-\hat{\epsilon}_{t,\alpha}f_0') + \hat{\mathcal{L}}_2(f_0\hat{1})\\ &= -\frac{f_0}{2}\{[\hat{\epsilon}_{t,\alpha},\hat{\Omega}^\alpha_{X_jQ_j}]_- - i(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_k},\hat{\Omega}^\alpha_{X_jQ_j}]\\ &\quad + i(\partial_{Q_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{X_k},\hat{\Omega}^\alpha_{X_jQ_j}]\}.\end{aligned} \tag{79}$$

Since $\hat{\epsilon}_{t,\alpha}$ is regarded as a small quantity, $-\hat{\epsilon}_{t,\alpha}f_0'$ is nothing but the $\mathcal{O}(\lambda)$ correction to the argument of f_0, i.e., $\hat{L}^{(0)}_{d,\alpha}$.

To set off also those terms in $\hat{\mathcal{L}}_2(f_0\hat{1})$ which are linear in f_0 itself, we need to introduce a correction to an *overall coefficient* of f_0, in addition to that to its argument. Namely, consider the following $\hat{f}_1$:

$$\hat{f}_1 \equiv -\hat{\epsilon}_{t,\alpha}f_0' - \frac{1}{2}\hat{\Omega}^\alpha_{X_jQ_j}f_0. \tag{80}$$

Then, when entering into $\hat{\mathcal{L}}_1$, the second term above totally cancels all the terms in the right hand side of Eq. (79).

Notice that this cancellation becomes possible only because the *relative ratio* among the coefficients of first three terms in $\hat{\mathcal{L}}_2(\hat{A}_\alpha)$ perfectly match with the ratio among the corresponding three terms in $\hat{\mathcal{L}}_1(\hat{A}_\alpha)$

$$\begin{aligned}\hat{\mathcal{L}}_2(\hat{A}_\alpha) = &-\frac{1}{4}[[\hat{\epsilon}_\alpha,\hat{\Omega}^\alpha_{X_j,Q_j}]_-,\hat{A}_\alpha]_+ - \frac{1}{4}[[\hat{\mathcal{M}}_\alpha,\hat{\Omega}^\alpha_{X_j,Q_j}]_-,\hat{A}_\alpha]_+\\ &+ \frac{i}{4}[(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_k},\hat{\Omega}^\alpha_{X_j,Q_j}],\hat{A}_\alpha]_+ - \{X_k\leftrightarrow Q_k\} + \cdots,\end{aligned}$$

$$\begin{aligned}\leftrightarrow\hat{\mathcal{L}}_1(\hat{A}_\alpha) = &-[\hat{\epsilon}_\alpha,\hat{A}_\alpha]_- - [\hat{\mathcal{M}}_\alpha,\hat{A}_\alpha]_- + i(\partial_{X_k}L^{(0)}_{d,\alpha})[\hat{D}^\alpha_{Q_k},\hat{A}_\alpha]\\ &- \{X_k\leftrightarrow Q_k\}.\end{aligned}$$

If either sign or coefficient of any one of the three terms in $\hat{\mathcal{L}}_2$ were not to meet with those of corresponding three terms in $\hat{\mathcal{L}}_1$, we could never have any $\hat{f}_1$ satisfying Eq. (76). In other words, there is no *a priori* guarantee for the existence of $\hat{f}_1$ which satisfies Eq. (76). In spite of this, both signs and coefficients in $\hat{\mathcal{L}}_2$ completely met with those in $\hat{\mathcal{L}}_1$, and we safely have Eq. (80) as an appropriate $\mathcal{O}(\lambda)$ correction to $\hat{A}_\alpha$. Reversely speaking, this perfect coincidence between $\mathcal{L}_1$ and $\mathcal{L}_2$ indicates the validity and the consistency of our derived SU(2) RKE.

C. Physical implications

Let us next argue the physical consequence of our solution $\hat{A}_\alpha$,

$$\hat{A}_\alpha = f_0(L^{(0)}_{d,\alpha})\hat{1} - \hat{\epsilon}_{t,\alpha}f_0'(L^{(0)}_{d,\alpha}) - \frac{1}{2}\hat{\Omega}^\alpha_{X_jQ_j}f_0(L^{(0)}_{d,\alpha}). \tag{81}$$

As was already mentioned, the second term above clearly stands for the renormalization of the energy dispersion. Namely, diagonalizing $\hat{\epsilon}_\alpha+\hat{\mathcal{M}}_\alpha\equiv\hat{\epsilon}_{t,\alpha}$, we can transcribe this correction into that of the arguments of f_0,

$$f_0\hat{1} - \hat{\epsilon}_{t,\alpha}f_0' \simeq \begin{bmatrix} f_0(L^{(0)}_{d,\alpha}-\epsilon_{t,\alpha,+}) & \\ & f_0(L^{(0)}_{d,\alpha}-\epsilon_{t,\alpha,-})\end{bmatrix}.$$

A finite $\epsilon_{t,\alpha,+}-\epsilon_{t,\alpha,-}$ thereby lifts the doubly degeneracy at equilibrium [Fig. 4(b)].

On the other hand, the last term in Eq. (81) changes the weight of the spectral function. To see this, let us integrate Eq. (81) with respect to frequency,

$$\int_{-\Lambda}^{\Lambda}\hat{A}_\alpha d\omega = Z^{(0)}_\alpha \times \left\{1 - \frac{1}{2}(\hat{\Omega}_{X_jQ_j})_{|\omega=\epsilon^{(0)}_\alpha}\right\}, \tag{82}$$

where $Z^{(0)}_\alpha$ stands for the residue at $L^{(0)}_{d,\alpha}=0$. This clearly dictates that, apart from the conventional renormalization factor $Z^{(0)}_\alpha$, the integrated spectral weight acquires an additional

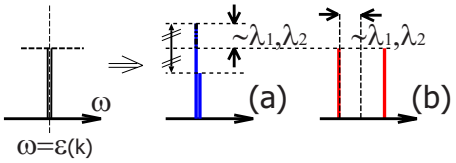

FIG. 4. (Color online) A schematic picture of spectral functions in SU(2) case. (a) The correction due to the Berry's curvature changes its weight. Furthermore, the change for the up-spin weight and that for the down-spin are, in general, different. (b) Usual first order correction change an energy spectrum.

$\mathcal{O}(\lambda)$ correction which is given by the SU(2) Berry's curvature.

This curvature correction $\hat{\Omega}^{\alpha}_{X_j Q_j}$ has a *spin-selective* effect. Namely, choosing the basis diagonalizing this SU(2) Berry's curvature, we have

$$\int_{-\Lambda}^{\Lambda} \hat{A}_{\alpha} d\omega \sim Z^{(0)}_{\alpha} \times \begin{bmatrix} 1-\frac{1}{2}\Omega^{\alpha,+}_{X_j Q_j} & 0 \\ 0 & 1-\frac{1}{2}\Omega^{\alpha,-}_{X_j Q_j} \end{bmatrix}_{|\omega=\epsilon^{(0)}_{\alpha}}, \tag{83}$$

where the up-chirality electron and the down one generally acquire the different corrections to its spectral weight with each other.

Observing Eq. (83), we can propose several photoemission experiments as a candidate experimental tool to visualize the dual SU(2) electromagnetic fields in a momentum-resolved way. To see this, consider the situation in the presence of applied electromagnetic fields, i.e., $\mathbf{b}=\nabla\times\mathbf{a}$ and $\mathbf{e}=\nabla a_0-\partial_T\mathbf{a}$. Then, using the canonical momentum and frequency introduced previously, Eq. (82) reduces into the *vector product* between the real electromagnetic fields and the a sort of dual version of SU(2) electromagnetic fields,

$$\begin{aligned} Z_{\alpha} &\equiv Z^{(0)}_{\alpha} \times \left\{ 1-\frac{1}{2}(\hat{\Omega}^{\alpha}_{X_j Q_j})_{|\omega=\epsilon^{(0)}_{\alpha}} \right\} \\ &= Z^{(0)}_{\alpha} \times \left\{ 1+\frac{1}{2}\bar{\mathcal{B}}^{\alpha}\cdot\mathbf{b}+\frac{1}{2}\bar{\mathcal{E}}^{\alpha}\cdot\mathbf{e} \right\}. \end{aligned} \tag{84}$$

Here, the dual quantities on the right hand side are estimated on shell at equilibrium,

$$\bar{\mathcal{B}}^{\alpha}_{m} = i\epsilon_{mnl}\{\partial_{k_n}\mathcal{A}^{\alpha}_{l}+\mathcal{A}^{\alpha}_{n}\mathcal{A}^{\alpha}_{l}\}_{|\omega=\epsilon^{(0)}_{\alpha}}, \tag{85}$$

$$\bar{\mathcal{E}}^{\alpha}_{m} = i\{\partial_0\mathcal{A}^{\alpha}_{m}-\partial_m\mathcal{A}^{\alpha}_{0}+[\mathcal{A}^{\alpha}_{0},\mathcal{A}^{\alpha}_{m}]\}_{|\omega=\epsilon^{(0)}_{\alpha}},$$

$$[\mathcal{A}^{\alpha}_{m}]_{\sigma\sigma'} \equiv \langle u^{(0),\alpha\sigma}_{k,\omega}|\partial_{k_m}u^{(0),\alpha\sigma'}_{k,\omega}\rangle,$$

$$[\mathcal{A}^{\alpha}_{0}]_{\sigma\sigma'} \equiv \langle u^{(0),\alpha\sigma}_{k,\omega}|\partial_{\omega}u^{(0),\alpha\sigma'}_{k,\omega}\rangle, \tag{86}$$

with $m=1,\ldots,d$.

As is clear from their coupling with real electromagnetic fields, $\bar{\mathcal{B}}^{\alpha}_{m}(k)$ and $\bar{\mathcal{E}}^{\alpha}_{m}(k)$ as functions of k obey precisely same symmetries as $\mathbf{M}_{\alpha,m}(k)$ and $\mathbf{P}_{\alpha,m}(k)$ do, respectively. Thus, in a SU(2) FL having time-reversal and spatial inversion symmetries, these SU(2) matrices become both traceless,

$$\mathrm{Tr}[\bar{\mathcal{E}}^{\alpha}_{m}(k)] \equiv \mathrm{Tr}[\bar{\mathcal{B}}^{\alpha}_{m}(k)] \equiv 0, \tag{87}$$

just as in Eq. (72). This suggests that the (integrated) spectral weights for the doubly degenerate quasiparticles, subjected under applied electromagnetic fields, acquire the first order corrections having a reverse sign with each other [see Fig. 4(a)].

The spectral weight for quasiparticle is, in principle, detectable in a momentum-resolved way in terms of photoemission experiments such as angle resolved photoemission spectroscopy (ARPES) and resonating inelastic x-ray scattering (RIXS). The above theoretical observation therefore leads us to raise the spin-resolved ARPES as the potential candidate tool to measure the dual SU(2) electromagnetic fields. Namely, observing the change of the spin-resolved spectral weight at each k point under small $\mathbf{e}$ or $\mathbf{b}$, we can identify the dual electric or magnetic field as its linear response. Furthermore, Eq. (87) claims that, maximizing the measured linear responses with respect to the resolved-spin's direction, one can even determine the spin-quantization axis for the eigenbasis diagonalizing $\bar{\mathcal{B}}^{\alpha}$ or $\bar{\mathcal{E}}^{\alpha}$. Thus, provided that these processes are performed for all the components of $\mathbf{e}$ and $\mathbf{b}$ separately, we could determine, as 2×2 matrices, all the $2\times d$ components of these SU(2) Berry's curvatures at an arbitrary k point on a Fermi surface.

The (spin-resolved) ARPES experiment under external electromagnetic fields is, however, a difficult experiment. A more relatively less unrealistic proposal might be detecting *Abelian* Berry's curvatures in U(1) FLs, such as ferromagnetic metals or paramagnetic metals without any centrosymmetric lattice point. In such U(1) FLs, we have only to regard all the SU(2) Hermitian matrices in Eqs. (67)–(69) as a scalar quantity to obtain the U(1) RKEs and its solutions up to the first order accuracy,[30]

$$\begin{aligned} A_{\alpha} &= \left(1-\frac{1}{2}\Omega^{\alpha}_{X_j Q_j}\right)\delta(L_{d,\alpha}-\mathcal{M}_{\alpha}) \\ &= \left(1+\frac{1}{2}\bar{\mathcal{B}}^{\alpha}\cdot\mathbf{b}+\bar{\mathcal{E}}^{\alpha}\cdot\mathbf{e}\right)Z_{\alpha}\delta(\omega-\epsilon_{\alpha}). \end{aligned} \tag{88}$$

Note that the dual Abelian electromagnetic fields are again defined to be on shell as in the SU(2) case,

$$\bar{\mathcal{B}}^{\alpha}_{j} \equiv i\epsilon_{jml}(\partial_{k_m}\mathcal{A}^{\alpha}_{l})_{|\omega=\epsilon_{\alpha}}, \quad \bar{\mathcal{E}}^{\alpha}_{j} \equiv i(\partial_{\omega}\mathcal{A}^{\alpha}_{j}-\partial_{j}\mathcal{A}^{\alpha}_{0})_{|\omega=\epsilon_{\alpha}},$$

$$\mathcal{A}^{\alpha}_{\mu} \equiv \langle u^{(0),\alpha}|\partial_{\mu}u^{(0),\alpha}\rangle, \tag{89}$$

with $\mu=\omega,k$.

In such a U(1) FL, the time-reversal (spatial inversion) counterpart being already lifted at equilibrium, these linear responses against $\mathbf{b}$ and $\mathbf{e}$ can be measured only by the spectral weight itself for a given Fermi surface. Namely, the spin filter is already implemented in matters themselves. Thereby, instead of ARPES experiments, the RIXS experiment, which is by far compatible with applied electromagnetic fields, is a more promising photoemission experiment.

V. U(1) EFFECTIVE BOLTZMANN EQUATION AND EQUATION OF MOTION FOR QUASIPARTICLES

In the previous section, we clarified the physical implications of the first four terms of $\hat{\mathcal{L}}_2$ given in Eq. (69). We will now study on quasiparticle dynamics, only to interpret the remaining four terms in $\hat{\mathcal{L}}_2$ such as

$$\hat{\mathcal{L}}_2 = \cdots - \frac{i}{2}(\partial_{X_k} L^{(0)}_{d,\alpha})[\hat{\Omega}^{\alpha}_{Q_k X_j}, [\hat{D}^{\alpha}_{Q_j}, \hat{A}_{\alpha}]] - \{X_j \leftrightarrow Q_j | X_k, Q_k\} - \{X_j, Q_j | X_k \leftrightarrow Q_k\} + \{X_j \leftrightarrow Q_j | X_k \leftrightarrow Q_k\}. \quad (90)$$

Strictly speaking, when it comes to an EOM for quasiparticles, we, in principle, have to begin with the Keldysh equation for the lesser (or greater) Green's functions $\hat{\mathsf{g}}^{<(>)}$, i.e.,

$$[G_0^{-1} - \Sigma^{\mathrm{HF}} - \sigma, \mathsf{g}^{<(>)}]_{\otimes,-} - [\Sigma_c^{<(>)}, \mathsf{b}]_{\otimes,-} = \frac{1}{2}[\Sigma_c^{>}, \mathsf{g}^{<}]_{\otimes,+} - \frac{1}{2}[\Sigma_c^{<}, \mathsf{g}^{>}]_{\otimes,+}, \quad (91)$$

which is different from that for the spectral function $\hat{\mathsf{A}}$.

As far as a system is not so far from its equilibration, however, this difference could be regarded as small at $T \simeq 0$ and $\omega \simeq \mu$. To be more specific, thanks to the boundary condition imposed on the Matsubara self-energy and its continuity, both lesser and greater collisional self-energies $\hat{\Sigma}_c^{<,>}(\omega)$ above become small around $\omega \simeq \mu$ (see the arguments in Appendies A 2 d and A 3). As a result, we could begin with the following dissipationless equation, instead of Eq. (91);

$$[\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}} - \hat{\sigma}, \hat{\mathsf{g}}^{<(>)}]_{\otimes,-} = \hat{0}. \quad (92)$$

Note that a small $\hat{\Sigma}_c^{<,>}(\omega)_{|\omega \simeq \mu}$ does not necessarily lead to a small $\hat{\sigma}(\omega)_{|\omega \simeq \mu}$. Namely, the latter one is given by the energy (principal) integral of the former two

$$\hat{\sigma}(\omega) \equiv \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\mathcal{P}}{\omega - \omega'} [\hat{\Sigma}_c^{<}(\omega') + \hat{\Sigma}_c^{>}(\omega')].$$

With the help of this biased treatment of the collisional self-energies for different frequency regions, however, we can obtain clear physical interpretations for all the remaining terms in $\hat{\mathcal{L}}_2$ given in Eq. (90). Namely, we can apply precisely the same projection process onto Eq. (92) as we did for $\hat{\mathsf{A}}$. Thus, we begin with the exactly same reduced Keldysh equation for $g_{\alpha}^{<}$ as in Eqs. (67)–(69), with $g_{\alpha}^{<}$ being the (α,α)th element of $N_b \times N_b$ matrix $\hat{g}^{<} \equiv \hat{U}^{\dagger} \hat{\mathsf{g}}^{<} \hat{U}$. By constructions, this diagonal element $g_{\alpha}^{<}$ can be decomposed into the product between the generalized Fermi distribution function f_{α} (scalar quantity) and the spectral function for the αth band, the latter of which was already derived in the previous section,[25]

$$g_{\alpha}^{<}(Q;X) = A_{\alpha}(Q;X) f_{\alpha}(Q;X). \quad (93)$$

Generally speaking, based on this decomposition, one must derive a coupled effective Boltzmann equation for spin and charge out of the SU(2) RKEs for $g_{\alpha}^{<}$. However, since this analysis is somehow involved, we will henceforth restrict ourselves to U(1) FLs, only to derive effective Boltzmann equation for charge degree of freedom, i.e., EOM only for f_{α}.

In the U(1) case, substituting Eq. (93) into the Abelian RKE, we first obtain

$$0 = \{\mathcal{L}_1(A_{\alpha}) + \mathcal{L}_2(A_{\alpha})\} f_{\alpha} + A_{\alpha}\{\partial_{X_j}(L_{d,\alpha} - \mathcal{M}_{\alpha})\partial_{Q_j} f_{\alpha} - \partial_{X_k}\Omega^{\alpha}_{Q_k X_j}\partial_{Q_j} f_{\alpha} + \partial_{Q_k}\Omega^{\alpha}_{X_k X_j}\partial_{Q_j} f_{\alpha} - (X_j \leftrightarrow Q_j)\}.$$

Notice that the spectral function A_{α} was determined such that $\mathcal{L}_1(A_{\alpha}) + \mathcal{L}_2(A_{\alpha}) \equiv 0$ (up to the second order in gradient expansion). Thus, in the right hand side, we can safely drop those terms which are proportional to f_{α} itself. Notice also that the spectral function A_{α} thus determined is sharply peaked at $\omega = \epsilon_{\alpha}$. Thereby, we have only to integrate this equation over frequency, such that f_{α} is replaced by the physical quasiparticle occupation number in the q-R space,

$$n_{\alpha}(q;R,T) \equiv f_{\alpha}(q, \omega \equiv \epsilon_{\alpha}; R, T).$$

The EOM thus obtained reads as follows:

$$[1 - \bar{\Omega}^{\alpha}_{T\epsilon_{\alpha}} + (\partial_{R_j}\epsilon_{\alpha})\bar{\Omega}^{\alpha}_{q_j\epsilon_{\alpha}} - (\partial_{q_j}\epsilon_{\alpha})\bar{\Omega}^{\alpha}_{R_j\epsilon_{\alpha}}]\partial_T n_{\alpha} = [\partial_{R_j}\epsilon_{\alpha} + (\partial_T\epsilon_{\alpha})\bar{\Omega}^{\alpha}_{\epsilon_{\alpha}R_j} - (\partial_{R_k}\epsilon_{\alpha})\bar{\Omega}^{\alpha}_{q_k R_j} + \bar{\Omega}^{\alpha}_{TR_j} + (\partial_{q_k}\epsilon_{\alpha})\bar{\Omega}^{\alpha}_{R_k R_j}]\partial_{q_j} n_{\alpha} - \{q_j \leftrightarrow R_j\},$$

which we can regard as the Boltzmann equation for the αth band quasiparticles. Note that the partial q, R, and T derivatives encoded into the curvatures therein apply only onto their *explicit* dependences and do not apply to their arguments of ϵ_{α}, e.g.,

$$\bar{\Omega}^{\alpha}_{q_i R_j} \equiv (\Omega^{\alpha}_{q_i R_j})_{|\omega=\epsilon_{\alpha}}, \quad \bar{\Omega}^{\alpha}_{\epsilon_{\alpha} R_j} \equiv (\Omega^{\alpha}_{\epsilon_{\alpha} R_j})_{|\omega=\epsilon_{\alpha}}.$$

As was explained in the Introduction, however, one can also introduce the curvature defined in the *codimensional space* associated with $\omega \equiv \epsilon_{\alpha}(q,R,T)$, e.g.,

$$\tilde{\Omega}^{\alpha}_{q_i R_j} \equiv i(\partial_{q_i}\tilde{\mathcal{A}}^{\alpha}_{R_j} - \partial_{R_j}\tilde{\mathcal{A}}^{\alpha}_{q_i}),$$

$$\tilde{\mathcal{A}}^{\alpha}_{R_j} \equiv \langle \tilde{u}^{\alpha} | \partial_{R_j} \tilde{u}^{\alpha} \rangle, \quad |\tilde{u}^{\alpha}\rangle \equiv |u^{\alpha}\rangle_{|\omega=\epsilon_{\alpha}}. \quad (94)$$

Then, the Boltzmann equation above can also be expressed solely in terms of these curvatures in the q-R-T space. Specifically, normalizing the coefficient of $\partial_T n_{\alpha}$, we obtain the following up to the second order in λ:

$$0 = \partial_T n_{\alpha} + \{\partial_{R_j}\epsilon + \tilde{\Omega}^{\alpha}_{TR_j} - (\partial_{R_k}\epsilon)\tilde{\Omega}^{\alpha}_{q_k R_j} + (\partial_{q_k}\epsilon)\tilde{\Omega}^{\alpha}_{R_k R_j}\}\partial_{q_j} n_{\alpha} - \{\partial_{q_j}\epsilon + \tilde{\Omega}^{\alpha}_{Tq_j} - (\partial_{R_k}\epsilon)\tilde{\Omega}^{\alpha}_{q_k q_j} + (\partial_{q_k}\epsilon)\tilde{\Omega}^{\alpha}_{R_k q_j}\}\partial_{R_j} n_{\alpha}. \quad (95)$$

Accordingly, comparing this Boltzmann equation with the continuity equation, i.e., $0 = \partial_T n_{\alpha} + (\partial_T q)\partial_q n_{\alpha} + (\partial_T R)\partial_R n_{\alpha}$, one can readily read the effective EOM for the quasiparticle,

$$\frac{dR_j}{dT} = -\partial_{q_j}\epsilon - \tilde{\Omega}^{\alpha}_{Tq_j} + (\partial_{R_k}\epsilon_{\alpha})\tilde{\Omega}^{\alpha}_{q_k q_j} - (\partial_{q_k}\epsilon_{\alpha})\tilde{\Omega}^{\alpha}_{R_k q_j},$$

$$\frac{dq_j}{dT} = \partial_{R_j}\epsilon + \tilde{\Omega}^{\alpha}_{TR_j} - (\partial_{R_k}\epsilon_\alpha)\tilde{\Omega}^{\alpha}_{q_k R_j} + (\partial_{q_k}\epsilon_\alpha)\tilde{\Omega}^{\alpha}_{R_k R_j}.$$

This effective EOM for quasiparticles proves, at least to the accuracy of second order in λ, that the EOM valid in "noninteracting" Fermi system[5] can also be generalized into "interacting" Fermi systems with those curvatures defined in the codimensional space, such as Eq. (94). However, observing that the first order correction to the renormalization factor is characterized by the Berry's curvature in Euclidean q-ω-R-T space instead of that in codimensional space [see Eq. (88)], studying the third order correction to this EOM is still an interesting open question (see also Sec. IV).

As an immediate application of this effective EOM, we can again consider the case with external electromagnetic fields,

$$\frac{dR}{dT} = \mathbf{v}_\alpha + \tilde{\mathcal{B}}^\alpha \times \frac{dk}{dT}, \tag{96}$$

$$\frac{dk}{dT} = -\mathbf{e} + \mathbf{b} \times \frac{dR}{dT}, \tag{97}$$

where $\mathbf{v}_\alpha \equiv \partial_k \epsilon_\alpha$ and $\tilde{\mathcal{B}}^\alpha$ reads

$$\tilde{\mathcal{B}}^\alpha \equiv \bar{\mathcal{B}}^\alpha - \bar{\mathcal{E}}^\alpha \times \mathbf{v}_\alpha. \tag{98}$$

Note that $\bar{\mathcal{B}}^\alpha$ and $\bar{\mathcal{E}}^\alpha$ are curvatures in Euclidean k-ω space [Eqs. (2) and (3) and Eqs. (5) and (6), respectively], while $\tilde{\mathcal{B}}^\alpha$ is the curvature defined in the codimensional space [Eqs. (9) and (10)]. Observing Eq. (96), one then see that the intrinsic anomalous Hall effect (AHE) in U(1) FLs should be defined in terms of both electromagnetic fields in the dual space,

$$\sigma_{jm} = \epsilon_{jml}\frac{e^2}{\hbar}\sum_\alpha \int \frac{dk}{(2\pi)^d}(\bar{\mathcal{B}}^\alpha - \bar{\mathcal{E}}^\alpha \times \mathbf{v}_\alpha)_l n_F(\epsilon_{\alpha,k}), \tag{99}$$

where n_F denotes a Fermi distribution function.

Notice that the term "intrinsic" is now generalized into a slightly wider sense. Namely, the above expression for the Hall conductivity contains not only the "k-space magnetic field" effect, which is already present in a noninteracting limit, but also the "many-body effect" via the corresponding "electric field" component. Based on the Ishikawa-Matsuyama formula and Fermi liquid assumptions, one can also see that this "many-body" correction indeed takes over (a part of) the so-called vertex correction to the static transverse conductivity (see Appendix B). This consistency check at the linear response regime strongly supports the validity of our derived effective EOM, i.e., Eqs. (96) and (97).

VI. SUMMARY AND OPEN PROBLEMS

To extract an information of low-energy effective theory in a generic *multiple-band* interacting Fermi systems, we have derived the reduced Keldysh equation (RKE) which effectively describes the charge and spin degrees of freedom for a specific *single-band* forming a Fermi surface. Our derivation is perturbative with respect to the gradient expansion whose coupling constant measures how a system is disequilibrated. It is, however, nonperturbative in the electron-electron interactions. Instead, it relies only on the "adiabatic assumption" which is also utilized to validate the usual Fermi liquid framework. This assumption claims that, when it comes to the low-frequency region at sufficiently low temperature, the intrinsic lifetime of a quasiparticle (due to electron-electron interactions), which is $\mathcal{O}(T^{-2},(\omega-\mu)^{-2})$, becomes much longer than the inverse of thermal broadening of the spectral functions [$\sim\mathcal{O}(T^{-1})$]. This assumption usually provides a finite energy region within which Fermi liquid theory works. Based on the same spirit and as a sort of zeroth order approximation, we completely ignored the lifetime part (anti-Hermitian part) of the collisional self-energies while studying on its Hermitian part on a general ground.

Out of the RKE thus derived, we have succeeded in extracting several intriguing physical implications both in SU(2) and U(1) FLs. A first observation is that the linear response of the spectral weight for a quasiparticle with respect to an applied electromagnetic field, $\mathbf{e}/\mathbf{b}$, is characterized by (what we call) the *dual electromagnetic fields*, i.e., $\bar{\mathcal{E}}^\alpha/\bar{\mathcal{B}}^\alpha$. Since the linear response condition of small $\mathbf{e}$ and $\mathbf{b}$ precisely coincides with the condition of validity of the gradient expansion, these linear response expressions are *asymptotically exact* for any quasiparticle on a Fermi surface at zero temperature. Based on this theoretical observation, we also tried to give some rough idea of how to measure the SU(2) Berry's curvatures in terms of photoemission experiments, so as to provide some future directions for the (spin) galvanomagnetic community (see Sec. IV).

Another important achievement is our derivation of the effective equation of motion (EOM) for a quasiparticle in a U(1) FL up to the second order accuracy in gradient expansion (see Sec. V). From this equation, one sees that both the k-space magnetic field and its corresponding temporal component, dubbed as electric field, contribute to the Lorentz force which acts on quasiparticles in k space. In the linear response regime, this effective EOM is indeed consistent with the exact many-body formula for the static transverse conductivity (see Appendix B).

We conclude with a discussion of future problems and issues. First, one ambiguity remains unresolved in our derivation of the reduced EOM. Specifically, working only to second order in the gradient expansion, we cannot exclude the possibility that $\mathbf{v}_\alpha$ in Eq. (98) may be more generally replaced by $\frac{dR}{dT}$. These two forms are equivalent up to second order but not more generally. To fix this ambiguity, a further analysis including third order effects in the gradient expansion would be necessary, which we leave for future work.

A more pressing issue is the well-controlled treatment of the anti-Hermitian part of the collisional self-energies. Having completely ignored the corresponding terms by hand, our results are valid, in a strict sense, only for those quasiparticles exactly on a Fermi surface and at zero temperature. Since any physical quantities do involve nonzero excitation of quasiparticles (even in linear response), this is clearly not satisfactory. While one might naïvely expect that collisional effects can be included by direct analogy to classical kinetics (e.g., in the relaxation time or more sophisticated approxima-

tions), the possibility of some more interesting interplay between collisions and the Berry phase physics captured herein cannot be excluded. This may be of particular interest if one considers the instabilities of conventional metals toward ordered states such as superconductors, dielectrics, etc. A treatment of both the anti-Hermitian part (decay of quasiparticles) and Hermitian part (adiabatic transport of quasiparticles) of the RKE on an equal footing is certainly warranted in the future.[31]

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APPENDIX A: BRIEF REVIEW OF KELDYSH EQUATION AND COLLISIONAL SELF-ENERGIES

To make this paper self-contained, we will briefly review the derivation of the Keldysh equation,[33] defining several kinds of self-energies introduced in Sec. II, such as $\hat{\Sigma}^{\mathrm{HF}}$, $\hat{\sigma}$, and $\hat{\Sigma}_c^{\langle,\rangle}$. In addition to this, we will also look into the boundary condition especially imposed on the collisional self-energy $\hat{\Sigma}_c^{\langle,\rangle}$. This boundary condition with additional arguments validates the dissipationless Keldysh equation for Green's functions, i.e., Eqs. (92) and (A26), starting from which we derived the U(1) effective Boltzmann equation in Sec. V.

In the next section, we will begin with the time-ordered Green's function defined on the imaginary-time domain $G(1,1')$ [see Eq. (A1) for its definition]. Specifically, the Dyson equation for this temperature (Matsubara) Green's function is derived first. The Hartree-Fock and collisional self-energy encoded there are also iteratively defined in terms of this Matsubara Green's function [see Eqs. (A5)–(A7)].

The time-ordered Green's function as a function of imaginary time t is analytic separately in two region: $\mathrm{Im}\, t \in [-\beta,0]$ and $[0,\beta]$. In other words, we can introduce two functions, usually dubbed as lesser and greater Green's functions, which coincide with this time-ordered one and which are analytic in these two regions, respectively,

$$G(1,1') = \begin{cases} G^{>}(1,1') & \text{for } \mathrm{Im}\, t_1 < \mathrm{Im}\, t_{1'} \\ -G^{<}(1,1') & \text{for } \mathrm{Im}\, t_{1'} < \mathrm{Im}\, t_1 \end{cases}$$

[compare Eq. (A1) with Eqs. (A8) and (A9)]. In the second section of this appendix, keeping the analyticity of these two, we will extend its time domain from the imaginary time into the real time domain. Correspondingly, the Dyson equation is also analytically continued onto the real-time domains [see Eqs. (A20), (A22), and (A23)].

We will see in the final section that periodic boundary condition imposed along the imaginary-time axis relates the lesser and greater collisional self-energies defined on the real-time domain with each other [see Eq. (A28)]. This relation guarantees that, at zero temperature and at equilibrium, lesser and greater self-energies have no weight at $\omega < \mu$ and $\omega > \mu$, respectively, i.e., Eqs. (A24) and (A25). These observations allow us to approximate the Keldysh equation derived further into the so-called dissipationless Keldysh equation [see the arguments from Eqs. (A22)–(A26)].

1. Dyson equation for temperature Green's function

Let us begin with the imaginary time Green's function defined in the "interaction" representations:

$$G(1,1';\phi,t_0) \equiv \frac{1}{i}\frac{\langle \mathcal{T}\{\hat{S}\psi(1)\psi^\dagger(1')\}\rangle}{\langle \mathcal{T}\{\hat{S}\}\rangle} = \frac{1}{i}\frac{\mathrm{Tr}[e^{-\beta(\mathcal{H}-\mu N)}\mathcal{T}\{\hat{S}\psi(1)\psi^\dagger(1')\}]}{\mathrm{Tr}[e^{-\beta(\mathcal{H}-\mu N)}\mathcal{T}\{\hat{S}\}]},$$

$$\hat{S} \equiv \exp\left[-i\int_{t_0}^{t_0-i\beta} d2\phi(2)\cdot\hat{n}(2)\right]. \tag{A1}$$

Note that the argument of the fermion operator j is an abbreviation of (r_j,t_j,α_j). Accordingly, $\int_{t_0}^{t_0-i\beta} dj$ includes not only the integral with respect to the imaginary time t_j but also the summation over the band indices α_j and the integral over the spatial coordinate,

$$\int_{t_0}^{t_0-i\beta} dj \equiv \sum_{\alpha_j}\int_{t_0}^{t_0-i\beta} dt_j \int dr_j.$$

$\mathcal{T}$ is the imaginary-time-ordering operator along $[t_0, t_0-i\beta]$, where t_0 is always real valued (in the next section, we will set t_0 to $-\infty$, after analytically continue t_1 and $t_{1'}$ onto the real-time domain),

$$\mathcal{T}\{\psi(1)\psi(1')\} \equiv \begin{cases} \psi(1)\psi(1') & \text{for } \mathrm{Im}\, t_1 < \mathrm{Im}\, t_{1'} \\ -\psi(1')\psi(1) & \text{for } \mathrm{Im}\, t_{1'} < \mathrm{Im}\, t_1. \end{cases}$$

The time dependence of the fermion operator $\psi(1) \equiv \psi_{\alpha_1}(r_1,t_1)$ is specified by $\mathcal{H} \equiv \mathcal{H}_0 + \mathcal{H}_1$:

$$-i\partial_{t_1}\psi(1) = [\mathcal{H},\psi(1)],$$

with $\mathcal{H}_0$ and $\mathcal{H}_1$ given in Eqs. (11) and (12). Then taking the t_1 derivative of Eq. (A1), we have the following EOM for this one-point Green's function,

$$\int_{t_0}^{t_0-i\beta} d\bar{1} G_0^{-1}(1,\bar{1}) G(\bar{1},1') = \delta(1-1') - i \int d2 V(1,2) G_2(1,2,1',2+)_{|t_2=t_1-i|\epsilon|}, \tag{A2}$$

where the inverse of a bare Green's function $G_0^{-1}(1,1')$ and two-point Green's function are defined as follows:

$$G_0^{-1}(1,1') \equiv [(i\partial_{t_1} - \phi(1))\delta_{\alpha_1\alpha_{1'}} - \hat{H}_0]\delta(1-1'),$$

$$G_2(1,2,1',2') \equiv -\frac{\langle \mathcal{T}\{\hat{S}\psi(1)\psi(2)\psi^\dagger(2')\psi^\dagger(1')\}\rangle}{\langle \mathcal{T}\{\hat{S}\}\rangle}.$$

"2+" in Eq. (A2) means that its temporal argument is chosen to be infinitesimally *later* than t_2 along the imaginary-time axis; $2+ \equiv (\alpha_2, r_2, t_2 - i|\epsilon|)$.

The auxiliary scalar potential $\phi(2)$ entering into Eq. (A1) as an interaction is utilized so as to describe the above two-point Green's function in terms of the self-energy Σ and one-point Green's function G,

$$\int_{t_0}^{t_0-i\beta} d\bar{1}[G_0^{-1}(1,\bar{1}) - \Sigma(1,\bar{1})]G(\bar{1},1') = \delta(1-1'). \tag{A3}$$

Notice first that the functional derivative of the one-point Green's function with respect to $\phi(2)$ brings about the two-point Green's function,

$$G_2(1,2,1',2+) = \left[G(2,2+) - \frac{\delta}{\delta\phi(2)}\right] G(1,1').$$

Thus, substituting this into Eq. (A2), we obtained a closed equation for the one-point Green's function,

$$\int_{t_0}^{t_0-i\beta} d\bar{1} G_0^{-1}(1,\bar{1}) G(\bar{1},1') = \delta(1-1') - i \int d2 V(1,2) \times \left[G(2,2+) - \frac{\delta}{\delta\phi(2)}\right]_{|t_2=t_1-i|\epsilon|} G(1,1'). \tag{A4}$$

This self-consistent equation can be readily transcribed into that for the self-energy Σ. Matrix multiplying Eq. (A4) by $G^{-1} \equiv G_0^{-1} - \Sigma$, we first obtain the following:

$$\Sigma(1,1') = -i\delta(1-1') \int d2 V(1,2) G(2,2+)_{|t_2=t_1-i|\epsilon|} - i\int_{t_0}^{t_0-i\beta} d\bar{1} \int d2 V(1,2) G(1,\bar{1}) \frac{\delta G^{-1}(\bar{1},1')}{\delta\phi(2)}_{|t_2=t_1-i|\epsilon|}.$$

Using $G^{-1} = G_0^{-1} - \Sigma$ and $\frac{\delta G_0^{-1}(1,1')}{\delta\phi(2)} = -\delta(1-1')\delta(1-2)$ on the right hand side above, the following iterative equation for the self-energy Σ is derived:

$$\Sigma(1,1') = -i\delta(1-1') \int d2 V(1,2) G(2,2+)_{|t_2=t_1-i|\epsilon|} + i\delta(t_1 - t_{1'}) V(1,1') G(1,1')_{|t_{1'}=t_1-i|\epsilon|} + i\int_{t_0}^{t_0-i\beta} d\bar{1} \int d2 V(1,2) G(1,\bar{1}) \frac{\delta\Sigma(\bar{1},1')}{\delta\phi(2)}_{|t_2=t_1-i|\epsilon|}. \tag{A5}$$

Being proportional to $\delta(t_1 - t_{1'})$, the first two terms are temporally instantaneous, which we usually call the Hartree-Fock term,

$$\Sigma^{\mathrm{HF}} \equiv -i\delta(1-1') \int d2 V(1,2) G(2,2+)_{|t_2=t_1-i|\epsilon|} + i\delta(t_1 - t_{1'}) V(1,1') G(1,1')_{|t_{1'}=t_1-i|\epsilon|}. \tag{A6}$$

A successive iteration through the third term of Eq. (A5) leads to the self-energy with higher order in V, which is clearly temporally noninstantaneous. We dub this noninstantaneous self-energy as the collisional self-energy henceforth,

$$\Sigma_c(1,1') \equiv \Sigma(1,1') - \Sigma^{\mathrm{HF}}(1,1'). \tag{A7}$$

Equation (A3) in combination with Eq. (A5) is the Dyson equation for the imaginary-time one-point Green's function.

2. Analytic continuations

a. Analytic continuation of $G^<$ and $G^>$

Real-time Green's functions are analytically continued from the following lesser and greater Green's functions defined on the imaginary-time domain,

$$G^>(1,1';\phi,t_0) = -i\frac{\langle \mathcal{U}(t_0,t_0-i\beta)\mathcal{U}^{-1}(t_0,t_1)\psi(1)\mathcal{U}(t_0,t_1)\mathcal{U}^{-1}(t_0,t_{1'})\psi^\dagger(1')\mathcal{U}(t_0,t_{1'})\rangle}{\langle \mathcal{U}(t_0,t_0-i\beta)\rangle}, \tag{A8}$$

$$G^<(1,1';\phi,t_0) = i\frac{\langle \mathcal{U}(t_0,t_0-i\beta)\mathcal{U}^{-1}(t_0,t_{1'})\psi^\dagger(1')\mathcal{U}(t_0,t_{1'})\mathcal{U}^{-1}(t_0,t_1)\psi(1)\mathcal{U}(t_0,t_1)\rangle}{\langle \mathcal{U}(t_0,t_0-i\beta)\rangle}. \tag{A9}$$

$\mathcal{U}(t,t') \equiv \mathcal{T}\{\exp[-i\int_t^{t'} d1\,\phi(1)\cdot\hat{n}(1)]\}$ characterizes the temporal evolution along the imaginary-time axis $[t_0, t_0 - i\beta]$ and, thus, is not a unitary operator.

Note that, at equilibrium and with $\phi \equiv 0$, these two functions are analytic in $\mathrm{Im}(t_1 - t_{1'}) \in [-\beta, 0]$ and $[0, \beta]$, respectively, regardless of $\mathrm{Re}\, t_1$ and $\mathrm{Re}\, t_{1'}$. Then, assuming that these analyticities also hold true in a weakly disequilibrated system with finite ϕ, we will introduce separately the lesser and greater Green's functions defined on the real-time domain as follows:

$$\mathsf{g}^{<}_{\alpha_1\alpha_{1'}}(r_1, r_{1'}{:}\mathrm{Re}\, t_1, \mathrm{Re}\, t_{1'}; \phi) \equiv \lim_{t_0 \to -\infty} G^{<}(1,1';\phi,t_0)_{|\mathrm{Im}\, t_1 = \mathrm{Im}\, t_{1'} + |\epsilon| = |\epsilon|},$$

$$\mathsf{g}^{>}_{\alpha_1\alpha_{1'}}(r_1, r_{1'}{:}\mathrm{Re}\, t_1, \mathrm{Re}\, t_{1'}; \phi) \equiv \lim_{t_0 \to -\infty} G^{>}(1,1';\phi,t_0)_{|\mathrm{Im}\, t_1 + |\epsilon| = \mathrm{Im}\, t_{1'} = 0}, \quad \text{(A10)}$$

by taking ϵ infinitesimally small. Choose the test scalar field $\phi(1)$ such that it vanishes at $\mathrm{Re}\, t_1 \to -\infty$. Then, $\mathcal{U}(t_0, t_0 - i\beta)$ appearing in both the denominator and the numerator of Eqs. (A8) and (A9) reduces to unity when $t_0 \to -\infty$. Accordingly, the two functions on the right hand side are solely defined on the real domain,

$$\mathsf{g}^{>}(1,1';\phi) = -i\langle \psi_\phi(1)\psi^\dagger_\phi(1')\rangle, \quad \text{(A11)}$$

$$\mathsf{g}^{<}(1,1';\phi) = i\langle \psi^\dagger_\phi(1')\psi_\phi(1)\rangle, \quad \text{(A12)}$$

with $\psi_\phi(1) \equiv \mathcal{U}^\dagger(-\infty, t_1)\psi(1)\mathcal{U}(-\infty, t_1)$. Now that $\mathcal{U}(-\infty, t)$ is a unitary operator, we can construct a real-time-ordered Green's function in terms of Eqs. (A11) and (A12),

$$\mathsf{g}(1,1';\phi) \equiv i\langle \mathcal{T}\{\psi_\phi(1)\psi_\phi(1')\}\rangle \equiv \begin{cases} \mathsf{g}^{>}(1,1';\phi) & \text{for } t_1 < t_{1'} \\ -\mathsf{g}^{<}(1,1';\phi) & \text{for } t_{1'} < t_1. \end{cases} \quad \text{(A13)}$$

b. Analytic continuation of $\hat{\Sigma}^{<}_c$ and $\hat{\Sigma}^{>}_c$

With the Dyson equation being composed also of the self-energy, let us next look into the analytic continuation of the collisional self-energy. The analytic property of the collisional self-energy can be obtained from that of Green's functions at equilibrium. At equilibrium, both the one-point Green's function $G(1,1')$ and the two-point Green's function $G_2(1,2,1',2+i|\epsilon|)_{|t_2 = t_1 - i|\epsilon|}$ are analytic in two regions, i.e., $\mathrm{Im}(t_1 - t_{1'}) \in [-\beta, 0]$ and $[0, \beta]$, separately. Compare these analyticities with the relation among the self-energy, one-point Green's function, and two-point Green's function,

$$\int_{t_0}^{t_0 - i\beta} d2\, V(1,2) G_2(1,2,1',2 + i|\epsilon|)_{|t_2 = t_1 - i|\epsilon|} \equiv \int_{t_0}^{t_0 - i\beta} d2\, \Sigma(1,2)\cdot G(2,1'). \quad \text{(A14)}$$

Then, one can readily see that $\Sigma_c(1,1')$ is also analytic both at these two regions, but separately. Thus, we can introduce the lesser and greater self-energies such that they are analytic in the following two regions, respectively,

$$\Sigma_c(1,1') \equiv \begin{cases} \Sigma^{>}_c(1,1') & \text{for } \mathrm{Im}\, t_1 < \mathrm{Im}\, t_{1'} \\ -\Sigma^{<}_c(1,1') & \text{for } \mathrm{Im}\, t_{1'} < \mathrm{Im}\, t_1. \end{cases}$$

In exactly the same way as we did for the one-point Green's function, we can then formally extend these two self-energies onto the real-time domain separately,

$$\Sigma^{<}_{c\,\alpha_1,\alpha_{1'}}(r_1, r_{1'}{:}\mathrm{Re}\, t_1, \mathrm{Re}\, t_{1'}; \phi) \equiv \lim_{t_0 \to -\infty} \Sigma^{<}_c(1,1';\phi,t_0)_{|\mathrm{Im}\, t_1 = \mathrm{Im}\, t_{1'} + |\epsilon| = |\epsilon|}$$

$$\Sigma^{>}_{c\,\alpha_1,\alpha_{1'}}(r_1, r_{1'}{:}\mathrm{Re}\, t_1, \mathrm{Re}\, t_{1'}; \phi) \equiv \lim_{t_0 \to -\infty} \Sigma^{>}_c(1,1';\phi,t_0)_{|\mathrm{Im}\, t_1 + |\epsilon| = \mathrm{Im}\, t_{1'} = 0}. \quad \text{(A15)}$$

c. Analytic continuation of Dyson equation

Now that both the self-energy and Green's functions are analytically continued onto the real-time domain, we will derive the Dyson equation for these real-time functions. We begin with that for the imaginary-time domain,

$$(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}) \cdot G^{<}(1,1') = \int_{t_0}^{t_0 - i\beta} d\bar{1}\, \Sigma_c(1,\bar{1}) G(\bar{1},1'). \quad \text{(A16)}$$

First decompose the right hand side into three terms, such that each term is expressed solely in terms of lesser and greater Green's function and collisional self-energy. Namely, depending on whether $t_{\bar{1}}$ locates within $[t_0, t_1]$, $[t_1, t_{1'}]$, or $[t_{1'}, t_0 - i\beta]$, the right hand side can be divided into the following three parts:

$$(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}) \cdot G^{<}(1,1') = \int_{t_0}^{t_1} d\bar{1}\, \Sigma^{>}_c(1,\bar{1}) G^{<}(\bar{1},1') + \int_{t_1}^{t_{1'}} d\bar{1}\, \Sigma^{<}_c(1,\bar{1}) G^{<}(\bar{1},1') + \int_{t_{1'}}^{t_0 - i\beta} d\bar{1}\, \Sigma^{<}_c(1,\bar{1}) G^{>}(\bar{1},1'), \quad \text{(A17)}$$

where the integral path is still along $[t_0, t_0 - i\beta]$ [see Fig. 5(a)]. Then, we will deform this integral path so that both t_1 and $t_{1'}$ reach the real-time axis. During this deformation, however, to keep the analyticity for all the Green's functions and self-energies in both the right and left hand sides, we have to observe the following condition:

$$-\beta < \mathrm{Im}\, t_{1'} < \mathrm{Im}\, t_1 < 0.$$

Due to this condition, for given "target" $\mathrm{Re}\, t_1$ and $\mathrm{Re}\, t_{1'}$, our integral path is uniquely deformed into an "L" shaped path depicted in Figs. 5(b) and 5(c),

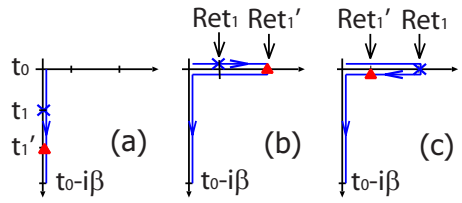

FIG. 5. (Color online) (a) The integral path for Eq. (A16). (b) Integral path for Eq. (A18) in the case of Re $t_1 <$ Re $t_{1'}$. (c) That for Re $t_1 >$ Re $t_{1'}$.

$$(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}) \cdot G^<(1,1') = \int_{t_0}^{t_1} d\bar{1}\Sigma_c^>(1,\bar{1})G^<(\bar{1},1') + \int_{t_1}^{t_{1'}} d\bar{1}\Sigma_c^<(1,\bar{1})G^<(\bar{1},1') + \int_{t_{1'}}^{t_0-i0} d\bar{1}\Sigma_c^<(1,\bar{1})G^>(\bar{1},1') + \int_{t_0-i0}^{t_0-i\beta} d\bar{1}\Sigma_c^<(1,\bar{1})G^>(\bar{1},1'). \quad \text{(A18)}$$

Note that the first three integral paths on the right hand side are all along the real-time domain, while the last one is strictly along $[t_0, t_0 - i\beta]$. So as to eliminate the final term, we will take t_0 to be $-\infty$. In this limit, $G^>(\bar{1},1';\phi)$ in the last term should vanish, since $|\mathrm{Re}\, t_{\bar{1}} - \mathrm{Re}\, t_{1'}| \to \infty$. This procedure simultaneously completes the analytic continuations of all the Green's functions and collisional self-energies encoded in this equation [see Eqs. (A10) and (A15)],

$$\begin{aligned}(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}) \cdot \mathsf{g}^<(1,1') &= \int_{-\infty}^{t_1} d\bar{1}\Sigma_c^>(1,\bar{1})\mathsf{g}^<(\bar{1},1') + \int_{t_1}^{t_{1'}} d\bar{1}\Sigma_c^<(1,\bar{1})\mathsf{g}^<(\bar{1},1') + \int_{t_{1'}}^{-\infty} d\bar{1}\Sigma_c^<(1,\bar{1})\mathsf{g}^>(\bar{1},1') \\ &= \int_{-\infty}^{t_1} d\bar{1}[\Sigma_c^>(1,\bar{1}) - \Sigma_c^<(1,\bar{1})]\mathsf{g}^<(\bar{1},1) - \int_{-\infty}^{t_{1'}} d\bar{1}\Sigma_c^<(1,\bar{1})[\mathsf{g}^>(\bar{1},1') - \mathsf{g}^<(\bar{1},1')]. \end{aligned} \quad \text{(A19)}$$

In a similar way, we can easily obtain an equation of motion for the greater Green's function,

$$\begin{aligned}(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}})\mathsf{g}^>(1,1') &= \int_{-\infty}^{t_1} d\bar{1}[\Sigma_c^>(1,\bar{1}) - \Sigma_c^<(1,\bar{1})]\mathsf{g}^>(\bar{1},1) \\ &\quad - \int_{-\infty}^{t_{1'}} d\bar{1}\Sigma_c^>(1,\bar{1})[\mathsf{g}^>(\bar{1},1') - \mathsf{g}^<(\bar{1},1')]. \end{aligned} \quad \text{(A20)}$$

At equilibrium, both Green's function and self-energy become translationally invariant in space and time, e.g.,

$$\mathsf{g}^{<(>)}(1,1';\phi \equiv 0) = \mathsf{g}^{<(>)}_{\alpha_1,\alpha_{1'}}(r_1 - r_{1'}, t_1 - t_{1'}),$$

$$\Sigma_c^{<(>)}(1,1';\phi \equiv 0) = \Sigma_c^{<(>)}{}_{\alpha_1,\alpha_{1'}}(r_1 - r_{1'}, t_1 - t_{1'}).$$

Thus, when Fourier transformed with respect to these relative coordinates, the convolutions appearing in Eq. (A20) would have reduced to a *simple product*, if the associated integral region were to range $[-\infty,\infty]$. In reality, however, their temporal integral regions do not range from $-\infty$ to $+\infty$. As a result, when Fourier transformed, they end up with several principal integrals with respect to the frequency. The *Hermitian* part of the collisional self-energy and Green's function *solely* take over these principal integral parts. To be specific, introduce the following two functions:

$$\mathsf{b}(1,1') \equiv \frac{1}{2}\frac{t_1 - t_{1'}}{|t_1 - t_{1'}|}[\mathsf{g}^>(1,1') - \mathsf{g}^<(1,1')],$$

$$\sigma(1,1') \equiv \frac{1}{2}\frac{t_1 - t_{1'}}{|t_1 - t_{1'}|}[\Sigma_c^>(1,1') - \Sigma_c^<(1,1')]. \quad \text{(A21)}$$

Then, we can formally rewrite the above equation in such a way that their convolutions with respect to time are always taken over $[-\infty,\infty]$,

$$(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}) \otimes \mathsf{g}^< = \sigma \otimes \mathsf{g}^< + \Sigma_c^< \otimes \mathsf{b} + \frac{1}{2}(\Sigma_c^> \otimes \mathsf{g}^< - \Sigma_c^< \otimes \mathsf{g}^>), \quad \text{(A22)}$$

$$(\hat{G}_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}) \otimes \mathsf{g}^> = \sigma \otimes \mathsf{g}^> + \Sigma_c^> \otimes \mathsf{b} + \frac{1}{2}(\Sigma_c^> \otimes \mathsf{g}^< - \Sigma_c^< \otimes \mathsf{g}^>), \quad \text{(A23)}$$

where we used the following abbreviated notations:

$$(A \otimes B)(1,1') \equiv \int_{-\infty}^{\infty} d\bar{1}A(1,\bar{1})B(\bar{1},1').$$

d. Dissipationless Keldysh equation

Equations (A22) and (A23) are what we call the Keldysh equation, from which we readily obtain Eq. (13).[32] Notice that the reexpression in terms of σ and b is not just for its formality but is, in fact, an important step to approximate this Keldysh equation based on physical arguments. As we see in the next section, at zero temperature and at equilibrium, the periodic boundary condition along the imaginary-time domain ensures that, when Fourier transformed with respect to their relative coordinates, the lesser (greater) collisional self-energy vanishes for those ω greater (less) than chemical potential μ, respectively,

$$\hat{\Sigma}_c^<(q,\omega) = \hat{0} \quad \text{for } \omega > \mu, \quad \text{(A24)}$$

$$\hat{\Sigma}_c^{>}(q,\omega)=\hat{0} \quad \text{for } \omega<\mu. \tag{A25}$$

Thereby, provided that a system is weakly disturbed from this equilibrated situation, we could still assume that these relations hold true for the collisional self-energy, with μ generalized into a function depending on R and T; $\mu\to\mu(R,T)$. Observing further the continuity of these collisional self-energies around $\omega\simeq\mu$, we could regard both of them to be sufficiently small quantities at these low frequency regions, i.e., $\omega\simeq\mu$.

The Hermitian part of the self-energy $\hat{\sigma}(\omega)$, however, is not necessarily a small quantity, even for $\omega\simeq\mu$. Namely, when Wigner transformed, those lesser and/or greater self-energies for high frequency regions do contribute to $\hat{\sigma}(\omega)$ at the low frequency region,[25]

$$\hat{\sigma}(q,R;\omega,T)=\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\mathcal{P}}{\omega-\omega'}(\hat{\Sigma}_c^{<}+\hat{\Sigma}_c^{>})(q,R;\omega',T).$$

Due to this different behavior between $\sigma(\omega)$ and $\Sigma_c^{<(>)}(\omega)$ around $\omega\simeq\mu$, we can approximate Eqs. (A22) and (A23) into the following dissipationless Keldysh equations:

$$(\hat{G}_0^{-1}-\hat{\Sigma}^{\mathrm{HF}}-\hat{\sigma})\otimes\hat{\mathfrak{g}}^{<(>)}=0, \tag{A26}$$

which is supposed to be still valid for $\omega\simeq\mu$ at sufficiently low temperature.

3. Periodic boundary conditions

Observing Eq. (A14), we will first read the boundary condition for the collisional self-energy, out of those for the one-point and two-point Green's functions. Notice that the one-point Matsubara Green's functions defined in Eqs. (A8) and (A9) obey the following boundary condition at equilibrium:

$$G^{<}(1,1')=-e^{\beta\mu}G^{>}(1-i\beta,1') \tag{A27}$$

for $-\beta<\mathrm{Im}\,t_{1'}<\mathrm{Im}\,t_1<0$ and arbitrary $\mathrm{Re}\,t_1$ and $\mathrm{Re}\,t_{1'}$. In a similar way, we can also see from its definition that the two-point Matsubara Green's function observes

$$G_2(1,2,1',2+i|\epsilon|)_{|t_2=t_1-i|\epsilon|}=-e^{\beta\mu}G_2(1-i\beta,2,1',2+i|\epsilon|)_{|t_2=t_1-i\beta-i|\epsilon|}$$

for $-\beta<\mathrm{Im}\,t_{1'}<\mathrm{Im}\,t_1<0$ and arbitrary $\mathrm{Re}\,t_1$ and $\mathrm{Re}\,t_{1'}$. When compared with Eq. (A14), these two then lead to the following boundary condition for the collisional self-energy:

$$\Sigma_c(1,1')=-e^{\beta\mu}\Sigma_c(1-i\beta,1').$$

While being imposed on the imaginary-time direction, this equation becomes useful when Fourier transformed with respect to $\mathrm{Re}\,t_1-\mathrm{Re}\,t_{1'}$,[25]

$$\Sigma_c^{<}(r_1,r_{1'};\omega)=e^{-\beta(\omega-\mu)}\Sigma_c^{>}(r_1,r_{1'};\omega). \tag{A28}$$

Namely, Eq. (A28) requires both of them to strictly observe Eqs. (A24) and (A25) at zero temperature, since either $\Sigma_c^{<}$ or $\Sigma_c^{>}$ cannot be singular in a usual metal.

APPENDIX B: CONSISTENCY WITH ISHIKAWA-MATSUYAMA-HALDANE FORMULA

With the conductivities being given by the current-current correlation functions, they are usually subject under the vertex corrections in interacting Fermi systems. However, when it comes to the static and transverse components, the Ward identity relates this vertex part with the derivative of the one-point Green's function, such that the Hall conductivity is expressed solely in terms of the single-particle Green's function

$$\sigma_\lambda=\frac{e^2}{2\hbar}\frac{\epsilon_{\lambda\mu\nu}}{(2\pi)^d}\int dk\int\frac{d\omega}{2\pi}e^{i\omega0+}\,\mathrm{Tr}\left[\frac{\partial\hat{\mathfrak{g}}}{\partial\omega}\frac{\partial\hat{\mathfrak{g}}^{-1}}{\partial k_\mu}\hat{\mathfrak{g}}\frac{\partial\hat{\mathfrak{g}}^{-1}}{\partial k_\nu}\right], \tag{B1}$$

which is known as the Ishikawa-Matsuyama formula.[11,22] Using only Fermi liquid assumptions, we will show in this appendix that our derived expression for the U(1) Hall conductivity, i.e., Eq. (99), is indeed consistent with this many-body formula. Namely, we will assume that, when diagonalized, each eigenvalue for the time-ordered one-point Green's function has a pole sufficiently closed to the real axis, such that the corresponding quasiparticle lifetime is infinitely long,

$$[\hat{g}_d]_{\alpha\alpha}\equiv[\hat{U}^{-1}\hat{\mathfrak{g}}\hat{U}]_{\alpha\alpha}=\frac{1}{\omega-E_{\alpha,k}(\omega)-i\,\mathrm{sgn}(\omega-\mu)\cdot0+}.$$

We can identify $\hat{U}$ above as the unitary matrix diagonalizing our Lagrangian $\hat{\mathfrak{L}}$.

Specifically, let us choose the basis in Eq. (B1) such that the Green's function is diagonalized,

$$\begin{aligned}\sigma_\lambda=\frac{e^2}{2\hbar}\frac{\epsilon_{\lambda\mu\nu}}{(2\pi)^d}\int dk\int\frac{d\omega}{2\pi}e^{i\omega0+}\,\mathrm{Tr}[&\{(\partial_\omega\hat{U})\hat{g}_d\hat{U}^{-1}\\&+\hat{U}(\partial_\omega\hat{g}_d)\hat{U}^{-1}+\hat{U}\hat{g}_d(\partial_\omega\hat{U}^{-1})\}\{(\partial_{k_\mu}\hat{U})\hat{g}_d^{-1}\hat{U}^{-1}\\&+\hat{U}(\partial_{k_\mu}\hat{g}_d^{-1})\hat{U}^{-1}+\hat{U}\hat{g}_d^{-1}(\partial_{k_\mu}\hat{U})^{-1}\}\hat{U}\hat{g}_d\hat{U}^{-1}\{(\partial_{k_\nu}\hat{U})\hat{g}_d^{-1}\hat{U}^{-1}\\&+\hat{U}(\partial_{k_\nu}\hat{g}_d^{-1})\hat{U}^{-1}+\hat{U}\hat{g}_d^{-1}(\partial_{k_\nu}\hat{U}^{-1})\}].\end{aligned} \tag{B2}$$

These $3^3\equiv27$ terms can be classified into three types, according to the matrix structure within the trace. To define these three types, note first that, in any single term among these 27 terms, $\hat{g}_d$ and $\hat{g}_d^{-1}$ clearly alternate each other within the trace. Moreover, between any pair of neighboring $\hat{g}_d$ and $\hat{g}_d^{-1}$, we have $\hat{U}$ and $\hat{U}^{-1}$ (or their derivative with respect to k or ω). When both of these $\hat{U}$ and $\hat{U}^{-1}$ are free from the derivatives, the corresponding $\hat{g}_d$ and $\hat{g}_d^{-1}$ can be directly connected. Meanwhile, when we have $\hat{U}^{-1}(\partial_Q\hat{U})\equiv\mathcal{A}_Q$ or $(\partial_Q\hat{U}^{-1})\cdot(\partial_{Q'}\hat{U})$ between a pair of neighboring $\hat{g}_d$ and $\hat{g}_d^{-1}$, they cannot be directly connected. We will first classify all the terms appearing within the above integrand in terms of the number of this "directly connected" pairs of $\hat{g}_d$ and $\hat{g}_d^{-1}$.

One class is those terms having no pair of directly connected Green's function and its inverse. For example, the following term belongs to this class:

$$\mathrm{Tr}[\hat{U}(\partial_\omega \hat{g}_d)\hat{U}^{-1}\hat{U}\hat{g}_d^{-1}(\partial_{k_\mu}\hat{U}^{-1})\hat{U}\hat{g}_d\hat{U}^{-1}(\partial_{k_\nu}\hat{U})\hat{g}_d^{-1}\hat{U}^{-1}]$$
$$= \mathrm{Tr}[(\partial_\omega \hat{g}_d^{-1})\hat{\mathcal{A}}_{k_\mu}\hat{g}_d\hat{\mathcal{A}}_{k_\nu}],$$

where one pair of $\hat{g}_d$ and $\hat{g}_d^{-1}$ was directly connected only to reduce into a unit, i.e., $\hat{g}_d\cdot\hat{g}_d^{-1}=1$. Such an *annihilated* pair is not regarded as a directly connected pair. Among the 27 terms enumerated above, there exist 16 terms belonging to this class, all of which can be summarized into several total derivatives,

$$\partial_{k_\nu}\{\mathrm{Tr}[\hat{\mathcal{A}}_\omega\hat{g}_d\hat{\mathcal{A}}_{k_\mu}\hat{g}_d^{-1}]\} + \partial_{k_\mu}\{\mathrm{Tr}[\hat{\mathcal{A}}_{k_\nu}\hat{g}_d\hat{\mathcal{A}}_\omega\hat{g}_d^{-1}]\}$$
$$+ \partial_\omega\{\mathrm{Tr}[\hat{\mathcal{A}}_{k_\nu}\hat{g}_d\hat{\mathcal{A}}_{k_\mu}\hat{g}_d^{-1}]\} \to 0.$$

Being contour integrated, all these terms vanish as indicated.

The second class is those terms having two directly connected pairs. Among 27 terms enumerated above, there clearly exists only one such a term,

$$\mathrm{Tr}[(\partial_\omega\hat{g}_d)(\partial_{k_\mu}\hat{g}_d^{-1})\hat{g}_d(\partial_{k_\nu}\hat{g}_d^{-1})\hat{g}_d(\partial_{k_\mu}\hat{g}_d^{-1})] = \mathrm{Tr}[(\partial_\omega\hat{g}_d)(\partial_{k_\nu}\hat{g}_d^{-1})\hat{g}_d(\partial_{k_\mu}\hat{g}_d^{-1})].$$

Then, observing the overall factor $\epsilon_{\lambda\mu\nu}$ in Eq. (B2), we can readily drop this term.

The third class consists of ten terms having only one directly connected pair. Four terms being canceled by one another, we have the following six terms remained;

$$-2\,\mathrm{Tr}[\hat{g}_d^{-1}(\partial_\omega\hat{g}_d)(\partial_{k_\mu}\hat{U}^{-1})(\partial_{k_\nu}\hat{U})]$$
$$-2\,\mathrm{Tr}[\hat{g}_d^{-1}(\partial_{k_\nu}\hat{g}_d)(\partial_\omega\hat{U}^{-1})(\partial_{k_\mu}\hat{U})]$$
$$+2\,\mathrm{Tr}[\hat{g}_d^{-1}(\partial_{k_\nu}\hat{g}_d)(\partial_{k_\mu}\hat{U}^{-1})(\partial_\omega\hat{U})] \quad \text{(B3)}$$

Notice that, when either the band index α or its momentum k denotes a *filled* Bloch state, the diagonalized time-ordered Green's functions has a pole at the upper half plane in the complex ω plane, say, $\omega=\epsilon_{\alpha,k}+i0+$,

$$[\hat{g}_d^{-1}(\partial_{k_\mu}\hat{g}_d)]_{\alpha\alpha} \equiv \frac{(\partial_{k_\mu}\epsilon_{\alpha,k})}{\omega-\epsilon_{\alpha,k}-i0+}, \quad \text{(B4)}$$

with $\epsilon_{\alpha,k}-E_{\alpha,k}(\epsilon_{\alpha,k})=0$. Note that, in the right hand side, we have assumed the infinite lifetime for these Bloch states. This assumption corresponds to ignoring the anti-Hermitian part of the collisional self-energy, i.e., $\hat{\Gamma}$.

Substituting Eq. (B3) into Eq. (B2), we then integrate over the frequency, such that the summation (integral) regions over $\alpha(k)$ are restricted within the filled Bloch states. Using further Eq. (B4), we then observe that the Hall conductivity is indeed characterized by the dual version of U(1) electromagnetic fields introduced in the text,

$$\sigma_\lambda = \frac{e^2}{\hbar}\frac{1}{(2\pi)^d}\sum_\alpha \int dk\{\bar{\mathcal{B}}^\alpha - (\bar{\mathcal{E}}^\alpha\times\mathbf{v}_\alpha)\}_\lambda,$$

$$\bar{\mathcal{B}}^\alpha_\lambda = i\epsilon_{\lambda\mu\nu}\{[(\partial_{k_\mu}\hat{U})^\dagger(\partial_{k_\mu}\hat{U})]_{\alpha\alpha}\}_{|\omega=\epsilon_{\alpha,k}},$$

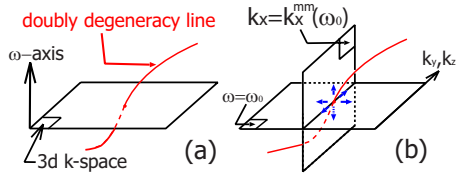

FIG. 6. (Color online) (a) $L_{d,\alpha}$ and $L_{d,\alpha+1}$ are degenerated at a line in the dual (3+1)-dimensional k space, which we dub as double degeneracy line (red curved line). The plane in this figure corresponds to a three-dimensional space, while its complementary axis is specified by the frequency ω. (b) $k_x=k_x^{mm}(\omega_0)$ plane and $\omega=\omega_0$ plane share a magnetic charge at $(\omega,k)=(\omega_0,k^{mm}(\omega_0))$. The relative sign between the charge viewed from the former and that from the latter is determined by the sign of the x component of magnetic charge current [compare Eqs. (C1) and (C5)].

$$\bar{\mathcal{E}}^\alpha_\lambda = i\{[(\partial_\omega\hat{U})^\dagger(\partial_{k_\lambda}\hat{U}) - \mathrm{c.c.}]_{\alpha\alpha}\}_{|\omega=\epsilon_{\alpha,k}},$$

with $\mathbf{v}_{\alpha,\lambda}=(\partial_{k_\lambda}\epsilon_{\alpha,k})$.

APPENDIX C: AMPERE'S LAW

In a noninteracting Fermi system, the Gauss's law solely determines the distribution of the U(1) magnetic field. When a double degeneracy point is formed by two neighboring energy dispersions, say αth and $(\alpha+1)$th bands at $k=k^{mm}$,

$$L_{d,\alpha}(k^{mm}) = L_{d,\alpha+1}(k^{mm}),$$

this degeneracy point in a three-dimensional k space becomes a source of the spatial divergence of the U(1) magnetic field associated with these two bands,

$$\nabla_k\cdot\bar{\mathcal{B}}^\alpha = \mathrm{sgn}\{\det V^\alpha\}\delta^3(k-k^{mm}) \equiv \rho_\alpha(k). \quad \text{(C1)}$$

The sign of the magnetic charge is given by the asymptotic form of the effective 2×2 Hamiltonian around this degeneracy point,

$$\hat{\mathrm{L}}^{(\alpha,\alpha+1)}(k) \simeq \sum_{\mu,\nu=x,y,z}(k_\mu-k_\mu^{mm})V^\alpha_{\mu\nu}\hat{\sigma}_\nu, \quad \text{(C2)}$$

where $\hat{\sigma}_\mu$ stands for the Pauli matrices.[33]

As was discussed in this paper, the eigenvalues of the Lagrangian $\hat{\mathrm{L}}$ in Fermi liquid acquire another argument ω in addition to the crystal momentum k. The doubly degeneracy point in a three-dimensional space, however, is by construction stable against any fourth axis. Thus, along this ω direction, this degeneracy point forms a degeneracy line [see Fig. 6(a)],

$$L_{d,\alpha}(k^{mm}(\omega),\omega) \equiv L_{d,\alpha+1}(k^{mm}(\omega),\omega).$$

Regarding ω as a sort of "time," one can therefore say that the U(1) magnetic charge in the three-dimensional k space is a conserved quantity. Corresponding to this conservation, we might as well introduce the U(1) magnetic monopole current,

$$j^{mm}_{\alpha,\mu}(k,\omega) \equiv \mathrm{sgn}\{\det V^{\alpha}\}\frac{dk^{mm}_{\mu}}{d\omega}\delta^{3}(k-k^{mm}(\omega)), \quad \text{(C3)}$$

so that its spatial divergence is balanced by the temporal derivative of the magnetic monopole density,

$$\sum_{\mu=x,y,z}\partial_{k_\mu} j^{mm}_{\alpha,\mu} + \partial_{\omega}\rho_{\alpha}(k,\omega) \equiv 0.$$

Then, in analogy to the Maxwell equation in a real space, we can introduce the dual version of "Ampere's law" for the U(1) electric field. Namely, the temporal derivative of the dual magnetic field and spatial rotation of the dual electric field is originated from this magnetic monopole current,

$$\nabla_k \times \bar{\mathcal{E}}^{\alpha} + \partial_{\omega}\bar{\mathcal{B}}^{\alpha} = -j^{mm}_{\alpha}. \quad \text{(C4)}$$

To see this law in a primitive way, one has only to consider the Gauss's law in another three-dimensional space, such as k_x=constant plane, which crosses the double degeneracy line mentioned above [see Fig. 6(b)]. For example, consider $k_x=k_x^{mm}(\omega_0)$ plane. This $\omega-k_y-k_z$ space contains a double degeneracy point at

$$(\omega,k_y,k_z) = (\omega_0,k_y^{mm}(\omega_0),k_z^{mm}(\omega_0)).$$

Therefore, the double degeneracy at $k=k^{mm}(\omega_0)$ also becomes a source of the "magnetic field" defined in this three-dimensional space, which is now $(\bar{\mathcal{B}}^{\alpha}_x,-\bar{\mathcal{E}}^{\alpha}_z,\bar{\mathcal{E}}^{\alpha}_y)$. To determine the sign of the corresponding "magnetic charge" viewed from this $\omega-k_y-k_z$ space, expand Eq. (C2) with respect to ω around $\omega=\omega_0$,

$$\sum_{\mu,\nu=x,y,z}\{k_\mu - k^{mm}_{\mu}(\omega)\}V^{\alpha}_{\mu\nu}(\omega)\hat{\sigma}_{\nu}$$

$$\simeq [\omega-\omega_0\ k_y-k_y^{mm}(\omega_0)\ k_z-k_z^{mm}(\omega_0)] \cdot \begin{bmatrix} -\frac{\partial k_x^{mm}}{\partial\omega} & -\frac{\partial k_y^{mm}}{\partial\omega} & -\frac{\partial k_z^{mm}}{\partial\omega} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{|\omega=\omega_0} \cdot [V^{\alpha}(\omega_0)] \cdot \begin{bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{bmatrix}.$$

Here, we imposed $k_x=k_x^{mm}(\omega_0)$ in the right hand side, only to discuss the Gauss's law in this plane. Namely, comparing this expression with Eqs. (C1) and (C2), we readily have the divergence of $(\bar{\mathcal{B}}^{\alpha}_x,-\bar{\mathcal{E}}^{\alpha}_z,\bar{\mathcal{E}}^{\alpha}_y)$ in this $\omega-k_y-k_z$ space,

$$\partial_{\omega}\bar{\mathcal{B}}^{\alpha}_x + \partial_{k_y}(-\bar{\mathcal{E}}^{\alpha}_z) + \partial_{k_z}\bar{\mathcal{E}}^{\alpha}_y = -\mathrm{sgn}\left\{\det V^{\alpha}\frac{\partial k_x^{mm}}{\partial\omega}\right\}_{|\omega=\omega_0}\delta(\omega-\omega_0) \times \delta(k_y-k_y^{mm}(\omega_0))\,\delta(k_z-k_z^{mm}(\omega_0)). \quad \text{(C5)}$$

By solving ω_0 in favor of k_x, we can actually see that this equation is nothing but the x component of Eq. (C4).

APPENDIX D: EXAMPLES OF ELECTRON-PHONON COUPLING SYSTEMS

In this appendix, based on a specific model calculations, we will present the crude estimate of the dual version of U(1) electric field. Basically, the electric component arises, in a perturbative calculation, from the frequency dependence of the self-energy. Moreover, we are interested in the Hermitian part of the self-energy. This is analytically more difficult to extract than well-studied anti-Hermitian part. Namely, the latter involves an energy-conserving delta function, while the former takes over the corresponding principal integrals. As a result, a consideration for the case of the Coulomb interaction becomes quite complex, the frequency dependence arising only at the two-loop level. We therefore consider instead the self-energy due to an electron-phonon interaction, for which we already have a non-trivial frequency dependence at the one-loop level. Though we did not discuss the electron-phonon interaction explicitly in our derivations, the considerations there still apply to this problem. Namely, our formulations are clearly free from the specific origin of the self-energy, provided that its anti-Hermitian part can be negligible at low frequency region.

As a simplest model, we considered the two-dimensional (2D) Rashba model defined on a square lattice, subjected to an external magnetic field along the z direction,

$$\mathcal{H}_0 = \frac{t}{2}\sum_{\langle jm\rangle,\alpha}\psi^{\dagger}_{j,\alpha}\psi_{m,\alpha} + h_z\sum_{j,\alpha,\beta}\psi^{\dagger}_{j,\alpha}[\hat{\tau}_z]_{\alpha,\beta}\psi_{j,\beta} - a\sum_j\sum_{\mu,\nu=x,y}\frac{i\epsilon_{\mu\nu}}{2}(\psi^{\dagger}_{j+e_\mu,\alpha}[\hat{\tau}_\nu]_{\alpha,\beta}\psi_{j,\beta} - \mathrm{H.c.}).$$

We employ the lattice-regularized model, since the Hermitian part of the self-energy usually depends on the ultraviolet cutoff. Namely, $\hat{\sigma}$ obtained from the Matsubara Green's function is composed of a principal integral with respect to momentum, which shows a ultraviolet divergence in a 2D continuum model. Although lattice regularized, the above Hamiltonian can be referred as a "Rashba" model in a sense that, when Fourier transformed, it reproduces the same energy dispersion as the standard (i.e., continuum) Rashba model around the Γ point,

$$\mathcal{H}_0 \equiv \sum_{\alpha,\beta}\sum_k \psi^{\dagger}_{\alpha}(k)[\hat{H}_0(k)]_{\alpha,\beta}\psi_{\beta}(k),$$

$$\hat{H}_0(k) \equiv -t(\cos k_x + \cos k_y - 2)\hat{1} + h_z\hat{\tau}_z + a\sin k_x\hat{\tau}_y - a\sin k_y\hat{\tau}_x,$$

with $\psi^{\dagger}_{\alpha}(k) \equiv \frac{1}{\sqrt{N}}\Sigma_j e^{ik\cdot j}\psi^{\dagger}_{\alpha}(j)$. Note also that a finite magnetic field h_z lifts the band degeneracy at $k=(0,0)$, $(\pi,0)$, $(0,\pi)$, and (π,π),

$$\mathsf{e}_{\pm,k} = -t(\cos k_x + \cos k_y - 2) \pm |\lambda_k|,$$

where $|\lambda_k| \equiv \sqrt{h_z^2 + a^2(\sin^2 k_x + \sin^2 k_y)}$. Concomitantly, the magnetic Bloch wave functions for these two energy bands also acquire finite z components of spin in the presence of h_z,

$$|\mathsf{u}_k^+\rangle = \begin{bmatrix} \cos\left(\frac{\theta_k}{2}\right) \\ \sin\left(\frac{\theta_k}{2}\right) e^{i\varphi_k} \end{bmatrix}, \quad |\mathsf{u}_k^-\rangle = \begin{bmatrix} -\sin\left(\frac{\theta_k}{2}\right) e^{-i\varphi_k} \\ \cos\left(\frac{\theta_k}{2}\right) \end{bmatrix},$$

where θ_k and φ_k are defined as follows:

$$(\cos\theta_k, \sin\theta_k) \equiv \frac{1}{|\lambda_k|}(h_z, a\sqrt{\sin^2 k_x + \sin^2 k_y}),$$

$$(\cos\varphi_k, \sin\varphi_k) \equiv \frac{1}{\sqrt{\sin^2 k_x + \sin^2 k_y}}(\sin k_x, \sin k_y).$$

We have introduced the magnetic field because, in a time-reversal symmetric system, the dual-electric-field contribution to the anomalous Hall effect (AHE), as well as that of the dual magnetic field, always vanishes. Namely, the electric field is time-reversally even in a T-invariant system,

$$\mathcal{E}^\alpha(k) = \mathcal{E}^\alpha(-k),$$

while the quasiparticle velocity is T-reversally odd,

$$\mathbf{v}_\alpha(k) = -\mathbf{v}_\alpha(-k).$$

Thus, $\mathcal{E}^\alpha(k) \times \mathbf{v}_\alpha(k)$ would vanish after the k integration in a T-invariant system.

As for many-body effects, we consider an electron-phonon (e-p) coupled Hamiltonian in which the Einstein phonon having constant mass $\omega_0 > 0$ interacts with this Rashba fermion,

$$\mathcal{H}_{\text{ph}} + \mathcal{H}_{e\text{-}p} = \sum_q \omega_0 b_q^\dagger b_q + \sum_{k,q,\alpha,\beta} \psi_\alpha^\dagger(k+q)[\hat{g}_q]_{\alpha,\beta}\psi_\beta(k) \cdot (b_q + b_{-q}^\dagger).$$

The Hermiteness requires $\hat{g}_q^\dagger = \hat{g}_{-q}$ in general. We take the simplest possible forms for this e-p coupling constant: $\hat{g}_q \simeq \hat{g}_{q=0}$ and $\hat{g}_{q=0} \simeq g \cdot \hat{1}$. As shown below, even employing this oversimplified coupling form, we already find a substantial magnitude of the dual electric field.

The lowest order (imaginary-time) self-energy associated with this electron-phonon system is the second order in g,

$$\hat{\Sigma}(k, i\omega_m) = \frac{g^2}{N}\sum_q \sum_{\gamma=\pm} |\mathsf{u}_{k+q}^\gamma\rangle\langle\mathsf{u}_{k+q}^\gamma| \left\{ \frac{n_B(\omega_0) + n_F(\mathsf{e}_{\gamma,k+q})}{i\omega_m + \omega_0 - \mathsf{e}_{\gamma,k+q}} + \frac{n_B(\omega_0) + 1 - n_F(\mathsf{e}_{\gamma,k+q})}{i\omega_m - \omega_0 - \mathsf{e}_{\gamma,k+q}} \right\}, \tag{D1}$$

where $n_B(\omega) \equiv \frac{1}{e^{\beta\omega}-1}$ denotes the Bose distribution function and N is the total number of lattice points.

At equilibrium, the lifetime part of the self-energy and Hermitian part of the self-energy $\hat{\sigma}$ are analytically continued from this imaginary-time one [see Eqs. (14) and (15)],

$$\Gamma_{\alpha\beta}(k,\omega) \equiv i\Sigma_{\alpha\beta}(k, i\omega_m = \omega + i|\delta|) - i\Sigma_{\alpha\beta}(k, i\omega_m = \omega - i|\delta|),$$

$$\sigma_{\alpha\beta}(k,\omega) \equiv \frac{1}{2}\{\Sigma_{\alpha\beta}(k, i\omega_m = \omega + i|\delta|) + \Sigma_{\alpha\beta}(k, i\omega_m = \omega - i|\delta|)\}.$$

The lifetime term clearly becomes zero at $T=0$ and $\omega \simeq \mu$. Namely, its integral region over the internal line is exponentially small:

$$[n_B(\omega_0) + n_F(\mathsf{e})] \cdot \delta(\omega + \omega_0 - \mathsf{e}) \simeq 0,$$

$$[n_B(\omega_0) + 1 - n_F(\mathsf{e})] \cdot \delta(\omega - \omega_0 - \mathsf{e}) \simeq 0,$$

when $|\omega - \mu| \ll \omega_0$ at $T \to 0$. On the other hand, the Hermitian part of the self-energy, which is made up of the principal integral, remains finite even at $T=0$,

$$\hat{\sigma}(k,\omega) = g^2 \sum_{\gamma=\pm} \left\{ \int_{\mathsf{e}_{\gamma,k'} \le \mu} \mathcal{P}\frac{|\mathsf{u}_{k'}^\gamma\rangle\langle\mathsf{u}_{k'}^\gamma|}{\omega + \omega_0 - \mathsf{e}_{\gamma,k'}} + \int_{\mathsf{e}_{\gamma,k'} \ge \mu} \mathcal{P}\frac{|\mathsf{u}_{k'}^\gamma\rangle\langle\mathsf{u}_{k'}^\gamma|}{\omega - \omega_0 - \mathsf{e}_{\gamma,k'}} \right\} \frac{dk'^2}{(2\pi)^2}, \tag{D2}$$

where we introduced the new integral variable $k' \equiv k+q$. Comparing Eq. (D2) with Eq. (D1), notice that this momentum k' is already taken to be dimensionless.

When integrated over k', the off-diagonal elements of the 2×2 matrix $|\mathsf{u}_{k'}^\gamma\rangle\langle\mathsf{u}_{k'}^\gamma|$ vanish. Namely, they are always odd functions of k_x' or k_y', while $\mathsf{e}_{\gamma,k'}$ is even,

$$|\mathsf{u}_{k'}^\gamma\rangle\langle\mathsf{u}_{k'}^\gamma| = \frac{1}{2}\hat{1} + \frac{\text{sgn}\,\gamma}{2}(\cos\theta_{k'}\hat{\tau}_z + \sin\theta_{k'}\cos\varphi_{k'}\hat{\tau}_x - \sin\theta_{k'}\sin\varphi_{k'}\hat{\tau}_y).$$

Accordingly, Eq. (D2) becomes diagonal,

$$\hat{\sigma}(k,\omega) = \{\cdots\}\cdot\hat{1} + \frac{g^2}{2}\sum_{\gamma=\pm}\left\{ \int_{\epsilon_{k',\gamma} \le \mu} \mathcal{P}\cdot\frac{\text{sgn}\,\gamma\cos\theta_{k'}}{\omega + \omega_0 - \epsilon_{k',\gamma}}\frac{dk'^2}{(2\pi)^2} + \int_{\epsilon_{k',\gamma} \ge \mu} \mathcal{P}\cdot\frac{\text{sgn}\,\gamma\cos\theta_{k'}}{\omega - \omega_0 - \epsilon_{k',\gamma}}\frac{dk'^2}{(2\pi)^2} \right\}\hat{\tau}_z \equiv S_0(\omega)\cdot\hat{1} + S_1(\omega)\hat{\tau}_z. \tag{D3}$$

The scalar function $S_1(\omega)$ is always a smooth function at $|\omega - \mu| \ll \omega_0$. When this self-energy diagonalized in combination with $\hat{H}_0(k)$, a nonzero derivative of $S_1(\omega)$ with respect to ω turns out to be indispensable for a finite dual electric field, which we will see below.

Working on a 2D model, we only have a single-component of magnetic field while two components of electric field,

$$(-\mathcal{E}_y^\gamma, \mathcal{E}_x^\gamma, \mathcal{B}_z^\gamma) \equiv \nabla \times \mathcal{A}^\gamma, \quad \mathcal{A}_\mu^\gamma \equiv i\langle u_{k,\omega}^\gamma|\partial_\mu u_{k,\omega}^\gamma\rangle,$$

where $\mu = (k_x, k_y, \omega)$ and $|u_{k,\omega}^\gamma\rangle$ diagonalizes the $\hat{\mathsf{L}}(k,\omega) = \hat{H}_0(k) + \hat{\sigma}(\omega)$. Being a two-band model, the Lagrangian can be decomposed in terms of the Pauli matrices,

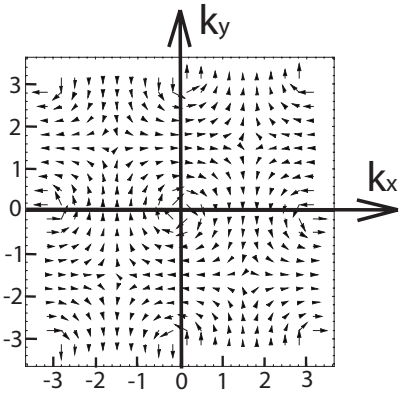

FIG. 7. A distribution of $(\mathcal{E}_x^-, \mathcal{E}_y^-)$ estimated at $\omega=\mu$. $\bar{a}=1.0$, $\bar{g}=2.0$, $\bar{h}_z=0.5$, $\bar{\omega}_0=0.5$, and $\mu=0$.

$$\hat{\mathsf{L}}(k,\omega) = \sum_{\mu=x,y,z} M_\mu(k,\omega)\cdot\hat{\tau}_\mu + \text{const}, \tag{D4}$$

where M now reads $(-a\sin k_y, a\sin k_x, h_z+S_1(\omega))$. Then, one readily see that the dual fields are identified as the solid angle subtended by this normalized vector $\hat{M}\equiv M/|M|$. For example, its spatial component reads

$$\begin{aligned}\mathcal{B}_z^\gamma &= -\frac{\text{sgn}\,\gamma}{2}(\nabla_{k_x}\hat{M}\times\nabla_{k_y}\hat{M})\cdot\hat{M}\\ &= \frac{\text{sgn}\,\gamma\cdot M_z}{2|M|^3}\cdot\frac{\partial M_y}{\partial k_x}\frac{\partial M_x}{\partial k_y} = -\frac{\text{sgn}\,\gamma\cdot a^2 M_z}{2|M|^3}\cos k_x\cos k_y,\end{aligned} \tag{D5}$$

with the band index $\gamma=\pm$. Similarly, the temporal components are given as follows:

$$\begin{aligned}(\mathcal{E}_x^\gamma,\mathcal{E}_y^\gamma) &= -\frac{\text{sgn}\,\gamma}{2|M|^3}\frac{\partial S_1}{\partial\omega}\hat{z}\cdot\left(\frac{\partial M}{\partial k_x}\times M,\frac{\partial M}{\partial k_y}\times M\right)\\ &= -\frac{\text{sgn}\,\gamma\cdot a^2}{2|M|^3}\frac{\partial S_1}{\partial\omega}(\cos k_x\sin k_y,-\cos k_y\sin k_x)\\ &\equiv \text{sgn}\,\gamma\cdot|E_k|(\cos k_x\sin k_y,-\cos k_y\sin k_x),\end{aligned}$$

which clearly indicates that a finite $\frac{\partial S_1}{\partial\omega}$ is the essential origin of the electric fields. With the lattice constant a_{lattice} being explicit, the magnitude of the dual electric field estimated on a Fermi surface, i.e., $\omega=\mu$, is given as follows:

$$|E_k|_{\omega=\mu} = \frac{a_{\text{lattice}}}{t}\cdot\frac{\bar{a}^2\bar{g}^2\bar{h}_z}{2|\bar{N}_k|^3}\times f,$$

$$f = \sum_{\gamma=\pm}\left\{\int_{\mathsf{e}_{\gamma,q}\le 0}\frac{\text{sgn}\,\gamma}{\bar{\Delta}_q\cdot(\bar{\omega}_0-\bar{\mathsf{e}}_{\gamma,q})^2} + \int_{\mathsf{e}_{\gamma,q}\ge 0}\frac{\text{sgn}\,\gamma}{\bar{\Delta}_q\cdot(\bar{\omega}_0+\bar{\mathsf{e}}_{\gamma,q})^2}\right\}\frac{dq^2}{(2\pi)^2}, \tag{D6}$$

where $\bar{\Delta}_q$ denotes the direct band gap at q point, measured with the transfer integral t being an energy unit; $\bar{\Delta}_q \equiv 2|\lambda_q|/t$. In a same sense, we also made it dimensionless, the Rashba coupling energy, e-p coupling energy, Zeeman energy, phonon energy, and band dispersion, like $\bar{a}$, $\bar{g}$, $\bar{h}_z$, $\bar{\omega}_0$, and $\bar{\mathsf{e}}_{\gamma,q}\equiv(\mathsf{e}_{\gamma,q}-\mu)/t$, respectively. In terms of these, the other dimensionless function $|\bar{N}_k|$ reads

$$|\bar{N}_k| = \sqrt{\bar{a}^2(\sin^2 k_y+\sin^2 k_x)+\bar{h}_z^2(1+\bar{S}_1)^2},$$

$$\bar{S}_1 = \bar{g}^2\sum_{\gamma=\pm}\left\{\int_{\mathsf{e}_{\gamma,q}\le 0}\frac{\text{sgn}\,\gamma}{\bar{\Delta}_q\cdot(\bar{\omega}_0-\bar{\mathsf{e}}_{\gamma,q})} - \int_{\mathsf{e}_{\gamma,q}\ge 0}\frac{\text{sgn}\,\gamma}{\bar{\Delta}_q\cdot(\bar{\omega}_0+\bar{\mathsf{e}}_{\gamma,q})}\right\}\frac{dq^2}{(2\pi)^2}.$$

Expression (D6) clearly demonstrates that the magnitude of the electric field is finite only in the presence of the applied magnetic field h_z and Rashba coupling a. It is also proportional to the dimensionless factor, i.e., f, whose value depends on a specific shape of the upper band and lower band in k space. From its form, however, there is no reason that this real-valued factor always has to reduce identically to zero. In fact, a simple numerical estimation shows that $f=-0.31$ and $\bar{S}_1=-0.24$, in the case of $\bar{a}=1.0$, $\bar{g}=2.0$, $\bar{h}_z=0.5$, and $\mu=0$ with $\bar{\omega}_0=0.5$. Observing these numerical factors, the magnitude of the electric field is estimated as follows:

$$|\mathcal{E}^\alpha| \simeq 10^{-1}\times\frac{a_{\text{lattice}}}{t}. \tag{D7}$$

The distribution of the dual electric field has a four-vortex structure within the unit cell $[-\pi,\pi]\times[-\pi,\pi]$ (see Fig. 7). These vortices reflect the four band-crossing points located at $(k_x,k_y,\omega)=(0,0,\omega_1)$, $(\pm\pi,\pm\pi,\omega_1)$, $(0,\pm\pi,\omega_1)$, and $(\pm\pi,0,\omega_1)$, with ω_1 defined as follows:

$$h_z + S_1(\omega_1) = 0.$$

Namely, $(-\mathcal{E}_y^-,\mathcal{E}_x^-,\mathcal{B}_z^-)$ has sources (2π charges) from the former two points, while it has sinks (-2π charges) at the latter two.

The many-body correction to the anomalous velocity is given by the outer product between the dual electric field and the quasiparticle velocity, i.e., $(\mathcal{E}_k^\alpha\times\mathbf{v}_\alpha)_z$. Thus, taking $|\mathbf{v}_\alpha|$ to be $a_{\text{lattice}}\cdot t$, we find it of the order of $10^{-1}\times a_{\text{lattice}}^2$. The bare contribution of the anomalous velocity on a Fermi surface can be directly estimated from Eq. (D5); $\mathcal{B}_z^\alpha\simeq 1.0\times a_{\text{lattice}}^2$. Observing these two quantities, we can then insist that the electric field contribution becomes almost at the same order of the dual-magnetic-field contribution and not negligible even in this oversimplified model.

*Present address: Division of Materials Physics, Department of Physical Science, Graduate School of Engineering Science, Osaka University. shindou@appi.t.u-tokyo.ac.jp

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[25] As in the standard literature (Ref. 23), we took the following conventions of the Fourier transformation of lesser and/or greater Green's function and/or collisional self-energy with respect to the frequency:

$$\mathsf{g}^{<}(\omega) = -i\int_{-\infty}^{\infty} e^{i\omega t}\mathsf{g}^{<}(t)dt, \quad \mathsf{g}^{>}(\omega) = i\int_{-\infty}^{\infty} e^{i\omega t}\mathsf{g}^{>}(t)dt,$$

$$\Sigma_c^{<}(\omega) = -i\int_{-\infty}^{\infty} e^{i\omega t}\Sigma_c^{<}(t)dt, \quad \Sigma_c^{>}(\omega) = i\int_{-\infty}^{\infty} e^{i\omega t}\Sigma_c^{>}(t)dt,$$

only to obtain the relation between the spectral function and Green's function given below,

$$\mathsf{A}(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t}\mathsf{A}(t)dt = \mathsf{g}^{>}(\omega) + \mathsf{g}^{<}(\omega)$$

[see also Eq. (17)].
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[28] Note that we have neglected the following terms which are of the order of $\mathcal{O}(\lambda^3)$:

$$[[\hat{\epsilon}_\alpha, \hat{\Omega}^\alpha_{X_j Q_j}]_+, \hat{A}_\alpha]_- \simeq \mathcal{O}(\lambda_1^2\lambda_2, \lambda_1\lambda_2^2).$$

Namely, $\hat{A}_\alpha$ being unit matrix at equilibrium, this commutator is third order in λ.
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[32] Note that the σ, b, and $G_0^{-1} - \hat{\Sigma}^{\mathrm{HF}}$ observe the following:

$$\sigma(1,1')^* \equiv \sigma(1',1), \mathsf{b}(1,1')^* \equiv \mathsf{b}(1',1),$$

$$(G_0^{-1} - \hat{\Sigma}^{\mathrm{HF}})(1,1')^* \equiv (G_0^{-1} - \hat{\Sigma}^{\mathrm{HF}})(1',1),$$

while

$$\mathsf{g}^{<(>)}(1,1')^* \equiv -\mathsf{g}^{<(>)}(1',1),$$

$$\Sigma_c^{<(>)}(1,1')^* \equiv -\Sigma_c^{<(>)}(1',1).$$

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